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Minimal almost convexity

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Abstract. In this article we show that the Baumslag–Solitar group $\mathrm{BS}(1,2)$ is minimally almost convex, or MAC. We also show that $\mathrm{BS}(1,2)$ does not satisfy Poénaru's almost convexity condition $P(2)$, and hence the condition $P(2)$ is strictly stronger than MAC. Finally, we show that the groups $\mathrm{BS}(1,q)$ for $q \geqslant 7$ and Stallings' non-FP_3 group do not satisfy MAC. As a consequence, the condition MAC is not a commensurability invariant.

1 Introduction

Let G be a group with finite generating set A, let Γ be the corresponding Cayley graph with the path metric d, and let $S(r)$ and $B(r)$ denote the sphere and ball, respectively, of radius r centered at 1 in Γ. The pair (G,A) satisfies the almost convexity condition AC_{f,r_0} for a function $f : \mathbb{N} \to \mathbb{R}_+$ and a natural number $r_0 \in \mathbb{N}$ if for every natural number $r \geqslant r_0$ and every pair of vertices $a, b \in S(r)$ with $d(a,b) \leqslant 2$, there is a path inside $B(r)$ from a to b of length at most $f(r)$. Note that every group satisfies the condition $\mathrm{AC}_{f,1}$ for the function $f(r) = 2r$. A group is *minimally almost convex*, or MAC (called $K(2)$ in [10]), if the condition AC_{f,r_0} holds for the function $f(r) = 2r - 1$ and some number r_0; that is, the least restriction possible is imposed on the function f. If the next least minimal restriction is imposed, i.e. if G is AC_{f,r_0} with the function $f(r) = 2r - 2$, then the group is said to be M′AC (called $K'(2)$ in [10]). Increasing the restriction on the function f further, the group satisfies Poénaru's $P(2)$ condition (see [7], [13]) if AC_{f,r_0} holds for a sublinear function $f : \mathbb{N} \to \mathbb{R}_+$, i.e. f satisfies the property that for every number $C > 0$ one has $\lim_{r\to\infty}(r - Cf(r)) = \infty$. All of these definitions are generalizations of the original concept of almost convexity given by Cannon in [3], in which the greatest restriction is placed on the function f, namely that a group is *almost convex* or AC if there is a constant function $f(r) \equiv C$ for which the group satisfies the condition $\mathrm{AC}_{C,1}$. Results of [3], [10], [14] show that the condition MAC, and hence each of

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the other almost convexity conditions, implies finite presentation of the group and solvability of the word problem.

The successive strengthenings of the restrictions in the definitions above give the implications AC $\Rightarrow P(2) \Rightarrow$ M$'$AC $\Rightarrow$ MAC. It is natural to ask which of these implications can be reversed. One family of groups to consider are the Baumslag–Solitar groups $\mathrm{BS}(1,q) := \langle a, t \mid tat^{-1} = a^q \rangle$ with $|q| > 1$, which Miller and Shapiro [12] proved are not almost convex with respect to any generating set.

In the present paper, the structure of geodesics in the Cayley graph of $\mathrm{BS}(1,q)$ is analyzed in greater detail, in Sections 2 and 3. In Section 3, we use this analysis to show that the group $\mathrm{BS}(1,2)$ satisfies the property M$'$AC. In Section 4 we show that the group $\mathrm{BS}(1,2)$ does not satisfy the $P(2)$ condition, and hence the implication $P(2) \Rightarrow$ M$'$AC cannot be reversed.

In Section 4 we also show that the groups $\mathrm{BS}(1,q) = \langle a, t \mid tat^{-1} = a^q \rangle$ for $q \geqslant 7$ are not MAC. Since the group $\mathrm{BS}(1,8)$ is a finite index subgroup of $\mathrm{BS}(1,2)$, an immediate consequence of this result is that both MAC and M$'$AC are not commensurability invariants, and hence not quasi-isometry invariants. The related property AC is also known to vary under quasi-isometry; in particular, Thiel [16] has shown that AC depends on the generating set.

Finally, in Section 5 we consider Stallings' non-FP_3 group [15], which was shown by the first author in [4], [5] not to be almost convex with respect to two different finite generating sets. In Theorem 5.3, we prove the stronger result that this group also is not MAC, with respect to one of the generating sets. Combining this with a result of Bridson [2] that this group has a quadratic isoperimetric function, we obtain an example of a group with quadratic isoperimetric function that is not MAC. During the writing of this paper, Belk and Bux [1] showed another such example; namely, they have shown that Thomson's group F, which also has a quadratic isoperimetric function function [9], does not satisfy MAC.

2 Background on Baumslag–Solitar groups

Let $G := \mathrm{BS}(1,q) = \langle a, t \mid tat^{-1} = a^q \rangle$ with generators $A := \{a, a^{-1}, t, t^{-1}\}$ for any natural number $q > 1$. Let Γ denote the corresponding Cayley graph with path metric d, and let $\mathscr{C}$ denote the corresponding Cayley 2-complex.

The complex $\mathscr{C}$ can be built from 'bricks' homeomorphic to $[0,1] \times [0,1]$, with both vertical sides labeled by a 't' pointing upward, the top horizontal side labeled by an 'a' to the right, and the bottom horizontal side split into q edges each labeled by an 'a' to the right. These bricks can be stacked horizontally into 'strips'. For each strip, q other strips can be attached at the top, and one on the bottom. For any set of successive upward choices, then, the strips of bricks can be stacked vertically to fill the plane. The Cayley complex then is homeomorphic to the Cartesian product of the real line with a regular tree T of valence $q+1$; see Figure 1. Let $\pi : \mathscr{C} \to T$ be the horizontal projection map. For an edge e of T, e inherits an upward direction from the upward labels on the vertical edges of $\mathscr{C}$ that project onto e. More details can be found in [6, pp. 154–160].

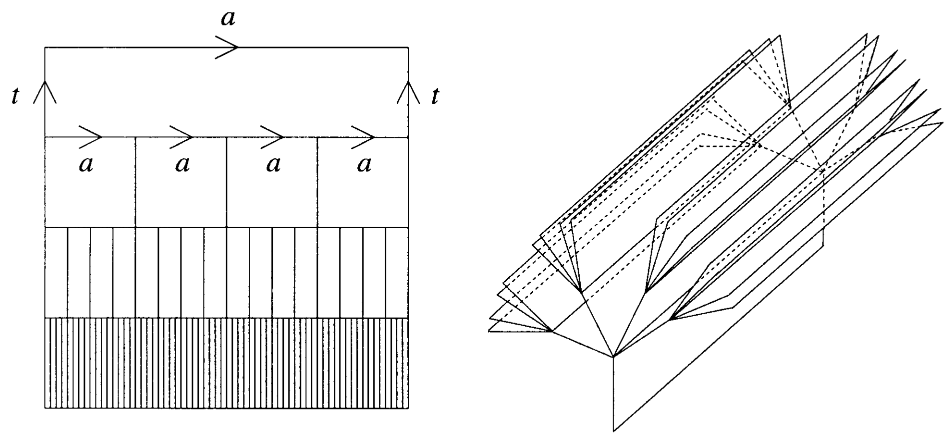

Figure 1. A brick in a plane, and a side-on view of the Cayley graph Γ for $\mathrm{BS}(1,4)$

For any word $w \in A^*$, let $\overline{w}$ denote the image of w in $\mathrm{BS}(1,q)$. For words $v, w \in A^*$, write $v = w$ if v and w are the same words in A^*, and $v =_G w$ if $\bar{v} = \overline{w}$. Let $l(w)$ denote the word length of w and let $w(i)$ denote the prefix of the word w containing i letters. Then $(w^{-1}(i))^{-1}$ is the suffix of w of length i. Define $\sigma_t(v)$ to be the exponent sum of all occurrences of t and t^{-1} in v. Note that the relator $tat^{-1}a^{-q}$ in the presentation of G satisfies $\sigma_t(tat^{-1}a^{-q}) = 0$; hence whenever $v =_G w$, then $\sigma_t(v) = \sigma_t(w)$.

The following lemma is well known; a proof can be found in [11].

Lemma 2.1 (Commutation). *If $v, w \in A^*$ and $\sigma_t(v) = 0 = \sigma_t(w)$, then $vw =_G wv$.*

Let E denote the set of words in $\{a, a^{-1}\}^*$, P the words in $\{a, a^{-1}, t\}^*$ containing at least one t letter, and N the words in $\{a, a^{-1}, t^{-1}\}^*$ containing at least one t^{-1} letter. A word $w = w_1 w_2$ with $w_1 \in N$ and $w_2 \in P$ will be referred to as a word in NP. Finally, let X denote the subset of the words in PN with t-exponent sum equal to 0. Letters in parentheses denote subwords that may or may not be present; for example, $P(X) := P \cup PX$. The following statement is proved in [8].

Lemma 2.2 (Classes of geodesics). *A word $w \in A^*$ that is a geodesic in Γ must fall into one of four classes*:

(1) *E or X*;

(2) *N or XN*;

(3) *P or PX*;

(4) *NP, or NPX with $\sigma_t(w) \geqslant 0$, or XNP with $\sigma_t(w) \leqslant 0$.*

Analyzing the geodesics more carefully, we find a normal form for geodesics in the following proposition.

Proposition 2.3 (Normal form). *If $w \in A^*$ is a geodesic in $G = \mathrm{BS}(1,q)$, then there is another geodesic $\hat{w} \in A^*$ with $\overline{\hat{w}} = \overline{w}$ such that for w in each class, $\hat{w}$ has the following form (respectively).*

(1) $\hat{w} = a^i$ *for* $|i| \leqslant C_q$ *where* $C_q := \left\lfloor \frac{q}{2} + 1 \right\rfloor$ *if* $q > 2$ *and* $C_2 := 3$, *or* $\hat{w} = w_0 \in X$.

(2) $\hat{w} = w_0 t^{-1} a^{m_1} \dots t^{-1} a^{m_e}$ *with* $|m_j| \leqslant \left\lfloor \frac{q}{2} \right\rfloor$ *for all* j, $e \geqslant 1$, *and either* $w_0 = a^i$ *for* $|i| \leqslant C_q$ *or* $w_0 \in X$.

(3) $\hat{w} = a^{n_0} t \dots a^{n_{f-1}} t w_0$ *with* $|n_j| \leqslant \left\lfloor \frac{q}{2} \right\rfloor$ *for all* j, $f \geqslant 1$, *and either* $w_0 = a^i$ *for* $|i| \leqslant C_q$ *or* $w_0 \in X$.

(4) *Either* $\hat{w} = t^{-e} a^{m_f} t a^{m_{f-1}} \dots a^{m_1} t w_0$ *with* $1 \leqslant e \leqslant f$, *or* $\hat{w} = w_0 t^{-1} a^{m_1} \dots t^{-1} a^{m_e} t^f$ *with* $1 \leqslant f \leqslant e$, *such that* $|m_j| \leqslant \left\lfloor \frac{q}{2} \right\rfloor$ *for all* j, *and either* $w_0 = a^i$ *for* $|i| \leqslant C_q$ *or* $w_0 \in X$. *Note that if* $\sigma_t(w) = 0$ *then* $e = f$ *and each expression is valid.*

In every class the word $w_0 \in X$ *can be chosen to be of the form either*

$$w_0 = t^h a^s t^{-1} a^{k_{h-1}} \dots a^{k_1} t^{-1} a^{k_0} \quad \text{or} \quad w_0 = a^{k_0} t a^{k_1} \dots t a^{k_{h-1}} t a^s t^{-h}$$

with $|k_j| \leqslant \left\lfloor \frac{q}{2} \right\rfloor$ *for all* j, $1 \leqslant |s| \leqslant q - 1$ *if* $q > 2$, $2 \leqslant |s| \leqslant 3$ *if* $q = 2$, *and* $h \geqslant 1$.

Proof. Note that for the natural number q, we have $q = \left\lfloor \frac{q}{2} + 1 \right\rfloor + \left\lceil \frac{q}{2} - 1 \right\rceil$.

For a geodesic w in class (1), if $w \in E$, then $w = a^i$ for some i. If $q = 2$, then $a^{\pm 6} = t a^{\pm 3} t^{-1}$ so that $|i| \leqslant 6$, and the words $a^{\pm(4+k)}$ have normal form $t a^{\pm 2} t^{-1} a^{\pm k} \in X$ for $k = 0$ and $k = 1$. If $q > 2$, then the relation $tat^{-1} =_G a^q$ can be reformulated as

$$a^{\pm(\lfloor \frac{q}{2}+1 \rfloor + 1)} =_G t a^{\pm 1} t^{-1} a^{\mp(\lceil \frac{q}{2}-1 \rceil - 1)}.$$

If q is even, then $a^{\pm(\frac{q}{2}+2)}$ is not geodesic, and so $|i| \leqslant \left\lfloor \frac{q}{2} + 1 \right\rfloor$. On the other hand, if q is odd, then

$$a^{\pm(\frac{q+1}{2}+2)} =_G t a^{\pm 1} t^{-1} a^{\mp(\frac{q-1}{2}-2)}$$

so that $a^{\pm(\frac{q+1}{2}+2)}$ is not geodesic; hence $|i| \leqslant \left\lfloor \frac{q}{2} + 1 \right\rfloor + 1$, and the words $a^{\pm(\lfloor \frac{q}{2}+1 \rfloor + 1)}$ have a normal form in X.

Next suppose that w is a geodesic in class (2). Then $w \in (X)N$, and so $w = w_0' t^{-1} a^{l_1} t^{-1} a^{l_2} \dots t^{-1} a^{l_e}$ for some word w_0' in class (1), $e \geqslant 1$, and integers l_i. Again we reformulate the defining relation of G, in this case to

$$t^{-1} a^{\pm \lfloor \frac{q}{2}+1 \rfloor} =_G a^{\pm 1} t^{-1} a^{\mp \lceil \frac{q}{2}-1 \rceil}.$$

If q is odd, then we may (repeatedly) replace any occurrence of $t^{-1} a^{\pm \lfloor \frac{q}{2}+1 \rfloor}$ by $a^{\pm 1} t^{-1} a^{\mp \lceil \frac{q}{2}-1 \rceil}$. If q is even then $t^{-1} a^{\pm \lfloor \frac{q}{2}+1 \rfloor}$ is not geodesic, and so $|l_j| \leqslant \left\lfloor \frac{q}{2} \right\rfloor$ for all j and replacements are not needed. In both cases, then, we obtain a geodesic word of the form $w_0''t^{-1}a^{m_1}t^{-1}a^{m_2}\dots t^{-1}a^{m_e}$ with each $|m_j| \leqslant \lfloor \frac{q}{2} \rfloor$ and w_0'' in class (1); to form the normal form $\hat{w}$, then, replace w_0'' by its normal form.

The normal form for geodesics in class (3) is established in a very similar way, using the relation $a^{\pm\lfloor \frac{q}{2}+1 \rfloor}t =_G a^{\mp\lceil \frac{q}{2}-1 \rceil}ta^{\pm 1}$.

Suppose next that w is a geodesic in class (4) with $\sigma_t(w) \geqslant 0$. Then

$$w = t^{-1}a^{k_1}\dots a^{k_{e-1}}t^{-1}a^{l_f}ta^{l_{f-1}}\dots a^{l_1}tw_0'$$

with w_0' in class (1), $1 \leqslant e < f$, and each $k_j, l_i \in \mathbb{Z}$. First we use Lemma 2.1 to replace w by the geodesic word

$$t^{-e}a^{l_f}ta^{k_{e-1}+l_{f-1}}\dots ta^{k_1+l_{f-e+1}}ta^{l_{f-e}}\tilde{w}_0ta^{l_{f-e-1}}\dots a^{l_1}tw_0'.$$

To complete construction of the normal form $\hat{w}$ from this word, we replace the subword $a^{l_f}t\dots a^{l_1}tw_0'$ by its normal form from class (3).

The constructions for the normal forms of geodesics w in class (4) with $\sigma_t(w) \leqslant 0$, and of geodesics $w_0 \in X$, are analogous.

3 The group BS(1, 2) satisfies M′AC

Let $G := \mathrm{BS}(1,2) = \langle a, t \mid tat^{-1} = a^2 \rangle$ with generators $A := \{a, a^{-1}, t, t^{-1}\}$. In this section we prove, in Theorem 3.5, that this group is M′AC. We begin with a further analysis of the geodesics in G, via several lemmas which are utilized in many of the cases in the proof of Theorem 3.5.

Lemma 3.1 (Large geodesic). *If w is a geodesic of length $r > 200$ in one of the classes* (1), (2) *or* (3) *of Proposition* 2.3 *and $|\sigma_t(w)| \leqslant 2$, then w is in X, XN or PX, respectively. Moreover, the X subword of w must have the form w_1w_2 with $w_1 \in P$ and $w_2 \in N$ such that $\sigma_t(w_1) = -\sigma_t(w_2) > 10$.*

Proof. Suppose that w is a geodesic in E, N or P of length $r > 200$, and $|\sigma_t(w)| \leqslant 2$. Then w contains at most two occurrences of the letters t and t^{-1}. As mentioned in the proof of Proposition 2.3, we have $a^{\pm 6} = ta^{\pm 3}t^{-1}$, and so a^j is not geodesic for $|j| \geqslant 6$. Hence w contains at most 15 occurrences of the letters a and a^{-1} interspersed among the $t^{\pm 1}$ letters. Then $l(w) \leqslant 17$, giving a contradiction.

Given a word $w_0 \in X$, there is a natural number $k \in \mathbb{N}$ with $w_0 =_G a^k$; write $\tilde{w}_0 := a^k$. If w is a geodesic word in $E \cup N \cup P \cup NP$, then let $\tilde{w} := w$. Combining these, for any geodesic word $w = w_0w_1$ (or $w = w_1w_0$) with $w_0 \in X$ and $w_1 \in N \cup P \cup NP$, define $\tilde{w} := \tilde{w}_0w_1 = a^kw_1$ (or $\tilde{w} := w_1\tilde{w}_0 = w_1a^k$, respectively). Then $\tilde{w} \in N \cup P \cup NP$, and the subword w_1 is geodesic.

Lemma 3.2. *If w is a word in NP, NPX or XNP, and $\tilde{w}$ contains a subword of the form $t^{-1}a^{2i}t$ with $i \in \mathbb{Z}$, then w is not geodesic.*

Proof. The word w can be written as $w = w_0 w_1 w_2$ with $w_1 \in NP$ and each of w_0 and w_2 in either X or E. Since $\tilde{w} = \tilde{w}_0 w_1 \tilde{w}_2$ contains the subword $t^{-1}a^{2i}t \in NP$, the word $t^{-1}a^{2i}t$ must be a subword of w_1, and hence also of w. Since $t^{-1}a^{2i}t =_G a^i$, this subword is not geodesic, and hence w also is not geodesic.

Lemma 3.3. *If w is any word in NP or NPN and $w =_G 1$, then w must contain a subword of the form $t^{-1}a^{2i}t$ for some $i \in \mathbb{Z}$.*

Proof. Since $G = \mathrm{BS}(1,2)$ is an HNN extension, Britton's Lemma yields that if $w \in NP(N)$ and $w =_G 1$, then w must contain a subword of the form ta^it^{-1} or $t^{-1}a^{2i}t$ for some $i \in \mathbb{Z}$. If $w \in NP$ then w must contain the second type of subword.

If $w \in NPN$, then $w = w_1 w_2 w_3$ with $w_1, w_3 \in N$ and $w_2 \in P$. Since $\sigma_t(w_1) < 0$ and $0 = \sigma_t(1) = \sigma_t(w_1) + \sigma_t(w_2) + \sigma_t(w_3)$, we have $\sigma_t(w_2) > \sigma_t(w_3)$ and $w_2 w_3 \in PX$. Then $w_2 w_3 = w_4 w_5$ with $w_4 \in P$ and $w_5 \in X$, and we have $w_1 w_4 \tilde{w}_5 \in NP$ with $w_1 w_4 \tilde{w}_5 =_G w =_G 1$. Now Britton's Lemma applies again to show that the prefix $w_1 w_4$ of w must contain a subword of the form $t^{-1}a^{2i}t$ for $i \in \mathbb{Z}$.

Lemma 3.4. *If w and u are geodesics, $w \in NP \cup XNP \cup NPX$, $\sigma_t(w) \leqslant \sigma_t(u)$, and $1 \leqslant d(\overline{w}, \bar{u}) \leqslant 2$, then $u \in NP \cup XNP \cup NPX$, and for some $w_1, u_1 \in N$ and $w_2, u_2 \in P$ with $\sigma_t(w_1) = \sigma_t(u_1)$, $\tilde{w} = w_1 w_2$ and $\tilde{u} = u_1 u_2$.*

Proof. The definition of $\tilde{w}$ shows that we can write $\tilde{w} = w_1 w_2$ with $w_1 \in N$ and $w_2 \in P$. Let γ label a path of length 1 or 2 from $\overline{w}$ to $\bar{u}$; since $\sigma_t(w) \leqslant \sigma_t(u)$, then $\gamma \in E \cup P$. Proposition 2.3 says that $\tilde{u} \in E \cup P \cup N \cup NP$. Since w is a geodesic, Lemma 3.2 implies that $\tilde{w}$ cannot contain a subword of the form $t^{-1}a^{2i}t$ for any integer i. Then Lemma 3.3 says that the word $\tilde{w}\gamma\tilde{u}^{-1}$, which represents the trivial element 1 in G, cannot be in $NP(N)$. Therefore $\tilde{u} \notin E \cup P \cup N$, and so $\tilde{u} \in NP$. Hence $u \in NP \cup XNP \cup NPX$.

We can now write $\tilde{u} = u_1 u_2$ with $u_1 \in N$ and $u_2 \in P$. The word $\tilde{u}^{-1}\tilde{w}\gamma = u_2^{-1}u_1^{-1}w_1 w_2 \gamma$ is another representative of 1. Repeatedly reducing subwords ta^jt^{-1} to a^{2j} in the subword $v := u_1^{-1}w_1 \in PN$ results in a word $\tilde{v} \in E \cup P \cup N$. Then $1 =_G u_2^{-1}\tilde{v}w_2\gamma \in NP(N)$, and so this word must contain a subword of the form $t^{-1}a^{2i}t$ for some integer i. Since w and u are geodesics, $w_1 w_2$ and $u_2^{-1}u_1^{-1}$ cannot contain such a subword. Therefore we must have $\tilde{v} \in E$. Hence $\sigma_t(w_1) = \sigma_t(u_1)$.

We split the proof of Theorem 3.5 into ten cases, depending on the classes from Proposition 2.3 to which the two geodesics w and u belong. In overview, we begin by showing that the first three cases cannot occur; that is, for a pair of geodesics w and u of length r in the respective classes in these three cases, it is not possible for $d(\overline{w}, \bar{u})$ to be less than 3. In Cases 4–6, we show that a path δ can be found that travels from $\overline{w}$ along the path w^{-1} to within a distance 2 of the identity vertex, and, after possibly traversing an intermediate edge, δ then travels along a suffix of u to $\bar{u}$. In Case 7 we show that the path δ can be chosen to have length at most 6, traveling around at most two bricks in the Cayley complex. In Case 8 there are subcases in which each of the two descriptions above occur, as well as a subcase in which the path δ initially follows the inverse of a suffix of w from $\overline{w}$, then travels along a path that 'fellow-travels' this initial subpath, and then repeats this procedure by traversing a fellow-traveler of a suffix of u, and then traveling along the suffix itself to $\bar{u}$. In Cases 9 and 10, the paths δ constructed in each of the subcases follow one of these three patterns.

Theorem 3.5. *The group* $G = \mathrm{BS}(1,2) = \langle a, t \mid tat^{-1} = a^2 \rangle$ *is* M′AC *with respect to the generating set* $A = \{a, a^{-1}, t, t^{-1}\}$. *In particular, if w and u are geodesics of length* $r > 200$ *with* $1 \leqslant d(\overline{w}, \bar{u}) \leqslant 2$, *then there is a path δ inside $B(r)$ from $\overline{w}$ to $\bar{u}$ of length at most* $2r - 2$.

Proof. Suppose that w and u are geodesics of length $r > 200$ with $1 \leqslant d(\overline{w}, \bar{u}) \leqslant 2$. Using Proposition 2.3, by replacing w and u by $\hat{w}$ and $\hat{u}$ respectively, we may assume that each of w and u is in one of the normal forms listed in that proposition. Using Lemma 3.1, we may assume that neither w nor u is in E.

Let γ be the word labeling a geodesic path of length at most 2 from $\overline{w}$ to $\bar{u}$, so that $w\gamma u^{-1} =_G 1$. Since $d(\overline{w}, \bar{u}) \geqslant 1$,

$$\gamma \in \{a^{\pm 1}, t^{\pm 1}, a^{\pm 2}, at^{\pm 1}, a^{-1}t^{\pm 1}, ta^{\pm 1}, t^{-1}a^{\pm 1}, t^{\pm 2}\}.$$

Then γ is in one of the sets E, P or N.

We divide the argument into ten cases, depending on the class of the normal forms w and u from Proposition 2.3, which we summarize in the following table.

Case	Class of w	Class of u	Case	Class of w	Class of u
Case 1	(4)	(1)	Case 6	(3)	(3)
Case 2	(4)	(3)	Case 7	(2)	(2)
Case 3	(2)	(3)	Case 8	(1)	(3)
Case 4	(1)	(1)	Case 9	(2)	(4)
Case 5	(1)	(2)	Case 10	(4)	(4)

This table represents a complete list of the cases to be checked. In particular, if w is in class (2) and u in class (1), then the inverse of the path in Case 5 will provide the necessary path δ, and similarly for the remainder of the cases.

Case 1: *w is in class* (4) *and u is in class* (1). Then w is in either NP, NPX or XNP, and $u \in X$. Since $\tilde{w} \in NP$, $\tilde{u} \in E$, and the path γ is in either E, N or P, then $1 =_G w\gamma u^{-1} =_G \tilde{w}\gamma\tilde{u}^{-1} \in NP(N)$ (that is, replacing the X subwords of w and u by powers of a). By Lemma 3.3, $\tilde{w}\gamma\tilde{u}^{-1}$ contains a subword of the form $t^{-1}a^{2s}t \in NP$, which therefore must occur within $\tilde{w}$. Then Lemma 3.2 says that w is not a geodesic, which is a contradiction. Hence Case 1 cannot arise.

Case 2: *w is in class* (4) *and u is in class* (3). Then w is in either NP, XNP or NPX, and $u \in P(X)$. In this case $1 =_G \tilde{w}\gamma\tilde{u}^{-1} \in NPN$, and the same proof as in Case 1 shows that Case 2 cannot occur.

Case 3: *w is in class* (2) *and u is in class* (3). Then $w \in (X)N$ and $u \in P(X)$. Since $\sigma_t(w) < 0$ and $\sigma_t(u) > 0$, we must have $\sigma_t(w) = -1$, $\sigma_t(u) = 1$, and $\gamma = t^2$. Lemma 3.1 says that $w \in XN$, and $u \in PX$. Since w is in normal form, $w = \hat{w} = w_0 t^{-1} a^i$ with $|i| \leqslant 1$ and $w_0 \in X$, and similarly $u = a^j t u_0$ with $|j| \leqslant 1$ and $u_0 \in X$. Then

$$1 =_G \tilde{w}\gamma\tilde{u}^{-1} =_G \tilde{w}_0 t^{-1} a^i t^2 \tilde{u}_0^{-1} t^{-1} a^{-j} \in NPN.$$

Lemma 3.3 then says that $w_0 t^{-1} a^i t^2 \tilde{u}_0^{-1} t^{-1} a^{-j}$ contains a subword of the form $t^{-1}a^{2s}t$ for some $s \in \mathbb{Z}$, so i must be a multiple of 2, and hence $i = 0$. Using the last part of Proposition 2.3, we can further write the normal form for $w_0 \in X$ as $w_0 = w_1 t^{-1}$, and so $w = w_i t^{-2}$. Then $u =_G w\gamma =_G w(r-2)$, contradicting the hypothesis that u is a geodesic word of length r. Thus Case 3 does not arise.

Case 4: *Both w and u are in class* (1). Then w and u are both in X. From Proposition 2.3 the normal forms $w = \hat{w}$ and $u = \hat{u}$ can be chosen of the form $\hat{w} = t^h w_1$ and $\hat{u} = t^i u_1$ with $h, i > 0$ and $w_1, u_1 \in N$. Then w and u have a common prefix $t = w(1) = u(1)$, and the path $\delta := w_1^{-1} t^{-(h-1)} t^{i-1} u_1$ from $\overline{w}$ through $\overline{w(1)}$ to $\bar{u}$ has length $2r - 2$ and stays inside $B(r)$.

Case 5: *w is in class* (1) *and u is in class* (2). Then $w \in X$ and $u \in (X)N$. In this case $\sigma_t(w) = 0$, $\sigma_t(\gamma) = \sigma_t(w^{-1}u) = \sigma_t(w) + \sigma_t(u) = \sigma_t(u)$, and $\sigma_t(u) < 0$, so that $\sigma_t(u)$ is either -1 or -2. The hypothesis that $r > 200$ and Lemma 3.1 imply that $u \in XN$. Then both of the normal forms $\hat{w}$ and $\hat{u}$ can be chosen to begin with t, and the same proof as in Case 4 gives the path δ.

Case 6: *Both w and u are in class* (3). In this case both w and u are in $P(X)$. Without loss of generality assume that $\sigma_t(w) \leqslant \sigma_t(u)$, and hence $\sigma_t(\gamma) \geqslant 0$ and $\gamma \in E \cup P$. Since both w and u are in normal form, $w = a^i t w_1 w_0$ and $u = a^j t u_1 u_0$ with $w_1, u_1 \in P \cup E$; $w_0, u_0 \in X \cup E$; and $|i|, |j| \leqslant 1$. Then

$$1 =_G u^{-1} w\gamma =_G \tilde{u}_0^{-1} u_1^{-1} t^{-1} a^{i-j} t w_1 \tilde{w}_0 \gamma \in NP.$$

By Lemma 3.3, $u_0^{-1} u_1^{-1} t^{-1} a^{i-j} t w_1 \tilde{w}_0 \gamma$ has a subword of the form $t^{-1}a^{2s}t$, so $i - j$ is a multiple of 2 and hence either $i = j$ with $0 \leqslant |i| \leqslant 1$ or $i = -j$ with $|i| = 1$.

If $i = j$ then w and u have a common prefix $a^i t = w(1 + |i|) = u(1 + |i|)$. The path $\delta := w_0^{-1} w_1^{-1} u_1 u_0$ from $\overline{w}$ follows the suffix $w_1 w_0$ of w backward to $\overline{w(1 + |i|)}$ and then follows the suffix $u_1 u_0$ of u to $\bar{u}$, remaining in $B(r)$.

If $i = -j$ with $|i| = 1$, define the path

$$\delta := w_0^{-1} w_1^{-1} a^{-i} u_1 u_0 =_G w_0^{-1} w_1^{-1} t^{-1} a^{-i} a^{-i} t u_1 u_0 = w^{-1} u =_G \gamma.$$

Then δ labels a path of length $2r-3$, traveling along w^{-1} from $\overline{w}$ to $\overline{w\delta(r-2)} = \overline{w(2)}$, then along a single edge to $\overline{w\delta(r-1)} = \overline{u(2)}$, and finally along a suffix of u to $\bar{u}$, thus remaining in $B(r)$. (See Figure 2.)

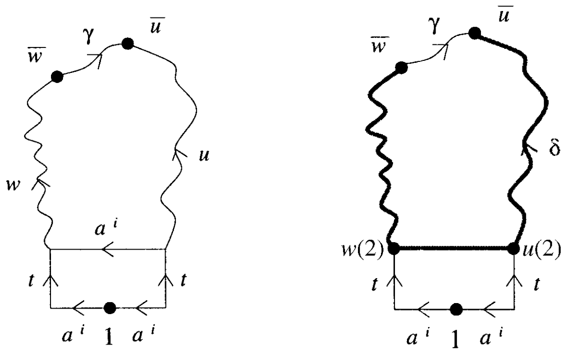

Figure 2. Case 6. $w = a^i t w_1 w_0$, $u = a^{-i} t u_1 u_0$

Case 7: *Both w and u are in class* (2). In this case, both w and u are in $(X)N$. We can assume without loss of generality that $\sigma_t(w) \leqslant \sigma_t(u)$, and so again $\sigma_t(\gamma) \geqslant 0$ and $\gamma \in E \cup P$.

From Proposition 2.3 we have $w = w_0 w_1 t^{-1} a^i$ and $u = u_0 u_1 t^{-1} a^j$ with $w_0, u_0 \in X \cup E$; $w_1, u_1 \in N \cup E$; and $|i|, |j| \leqslant 1$. Thus

$$1 =_G \tilde{w}\gamma\tilde{u}^{-1} = \tilde{w}_0 w_1 t^{-1} a^i \gamma a^{-j} t u_1^{-1} \tilde{u}_0^{-1} \in NP.$$

By Lemma 3.3 the latter contains a subword of the form $t^{-1}a^{2s}t$, and so $t^{-1}a^i\gamma a^{-j}t$ must contain this subword.

Since $\gamma \in E \cup P$, we have $\gamma \in \{t, t^2, ta^{\pm 1}, a^{\pm 1}t, a^{\pm 1}, a^{\pm 2}\}$, and we may divide the argument into four subcases.

Case 7.1: $\gamma \in \{t, a^{\pm 1}t\}$. Then $t^{-1}a^{2s}t$ must be a subword of $t^{-1}a^i\gamma$. If $\gamma = t$, then since $|i| \leqslant 1$ we have $i = 0$ and $w = w_0 w_1 t^{-1}$, hence

$$u =_G w\gamma =_G w(r-1).$$

If $\gamma = a^{\pm 1}t$, then $|i| = 1$, and $\gamma = a^{\pm i}t$. If $\gamma = a^i t$, then

$$u =_G w\gamma = w(r-2)t^{-1}a^i a^i t =_G w(r-2)a^i.$$

Finally, if $\gamma = a^{-i}t$, then

$$u =_G w\gamma = w(r-2)t^{-1}a^i a^{-i} t =_G w(r-2).$$

Each of these three options results in a contradiction to the fact that u is a geodesic of length r, and so Subcase 7.1 cannot occur.

Case 7.2: $\gamma \in \{t^2, ta^{\pm 1}\}$. In this subcase, $t^{-1}a^{2s}t$ must be a subword of $t^{-1}a^i t$ again, and so $i = 0$ and $w = w_0 w_1 t^{-1}$. Note that $\gamma(1) = t$ and $w\gamma(1) =_G w(r-1)$. Then γ is a path of length 2 inside $B(r)$ from w to u. In this subcase, we may define the path $\delta := \gamma$.

Case 7.3: $\gamma \in \{a^{\pm 1}\}$. Write $\gamma = a^k$ with $|k| = 1$. Recall that $0 \leqslant |i| \leqslant 1$.

If $i = 0$, then $t^{-1}a^{2s}t$ must be a subword of $t^{-1}a^k a^{-j} t$, hence $2 \mid (k - j)$ and $|j| = 1$. Then $\gamma = a^{\pm j}$. If $\gamma = a^j$, then

$$w =_G u\gamma^{-1} = u(r-2)t^{-1}a^j a^{-j} =_G u(r-2)t^{-1},$$

contradicting the length r of the geodesic w. Thus $\gamma = a^{-j}$. The word $\delta := ta^{-j}t^{-1}a^j =_G a^{-j}$ labels a path from $\bar{w}$ to $\bar{u}$ of length 4. Since $w\delta(1) =_G w(r-1)$ and $w\delta(2) =_G u(r-2)$, the path δ stays inside $B(r)$, and hence satisfies the required properties. (See Figure 3.)

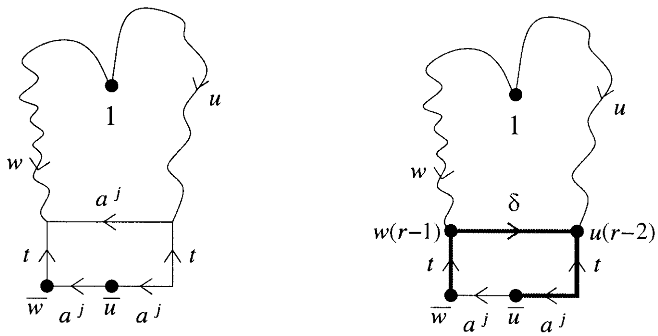

Figure 3. Case 7.3. $w = w_0 w_1 t^{-1}$, $u = u_0 u_1 t^{-1} a^j$

If $|i| = 1$, then we can write $\gamma = a^{\pm i}$. If $\gamma = a^{-i}$, then $u =_G w\gamma =_G w(r-1)$, again giving a contradiction; hence $\gamma = a^i$. Note that the word $t^{-1}a^{2s}t$ must be a subword of $t^{-1}a^{2i}a^{-j}t$, hence $2 \mid (2i - j)$ and $j = 0$. Write $\delta := a^{-i}ta^i t^{-1} =_G a^i$; then δ labels a path of length 4 from $\bar{w}$ to $\bar{u}$, with $w\delta(2) =_G w(r-2)$ and $w\delta(3) =_G u(r-1)$, and so the path remains in $B(r)$ as required.

Case 7.4: $\gamma \in \{a^{\pm 2}\}$. Write $\gamma = a^{2k}$ with $|k| = 1$. As in the previous subcase, we consider the options $i = 0$ and $|i| = 1$ in separate paragraphs.

If $i = 0$, then $t^{-1}a^{2s}t$ must be a subword of $t^{-1}a^{2k}a^{-j}t$, and so $j = 0$. Then the path of length 3 labeled by $\delta := ta^k t^{-1}$ from $\bar{w}$ to $\bar{u}$ traverses the vertices represented by $w\delta(1) =_G w(r-1)$ and $w\delta(2) =_G u(r-1)$, hence remaining in $B(r)$.

If $|i| = 1$, then $\gamma = a^{\pm 2i}$. If $\gamma = a^{-2i}$, then $w\gamma(1) =_G w(r-1)$, and so we may define $\delta := \gamma$.

If $|i| = 1$ and $\gamma = a^{2i}$, then $t^{-1}a^{2s}t$ must be a subword of $t^{-1}a^{3i}a^{-j}t$. Thus $|j| = 1$, and so $j = \pm i$. If $j = i$, then $w\gamma(1) =_G u(r-1)$, hence again the path $\delta := \gamma$ has the required properties. If $j = -i$, then the path of length 6 labeled by $\delta := a^{-i}ta^{2i}t^{-1}a^{-i} =_G a^{2i}$ starting at $\overline{w}$ ends at $\bar{u}$. Since $w\delta(2) =_G w(r-2)$ and $w\delta(4) =_G u(r-2)$, this path also remains within $B(r)$.

Case 8: *w is in class* (1) *and u is in class* (3). Then $w \in X$ and $u \in P(X)$. In this case $\sigma_t(w) = 0$, $\sigma_t(u) > 0$, and $\sigma_t(u) = \sigma_t(w) + \sigma_t(\gamma) = \sigma_t(\gamma)$, so that $0 < \sigma_t(u) = \sigma_t(\gamma) \leqslant 2$. Thus $\gamma \in P$, and so $\gamma \in \{t, ta^{\pm 1}, t^2, a^{\pm 1}t\}$.

Suppose that $\gamma \in \{t, ta^{\pm 1}, t^2\}$. By Proposition 2.3 and Lemma 3.1, the normal form w can be chosen in the form $w = w_1 t^{-h}$ with $w_1 \in P$ and $h > 10$. Then the geodesic u of length r cannot represent $wt =_G w(r-1)$ or $wt^2 =_G w(r-2)$, thus $\gamma \neq t$ and $\gamma \neq t^2$. For $\gamma = ta^{\pm 1}$, since $w\gamma(1) =_G w(r-1)$, we may define $\delta := \gamma$.

Suppose for the rest of Case 8 that $\gamma = a^{\pm 1}t$ and write $\gamma = a^m t$ with $|m| = 1$. Proposition 2.3 says that the normal form w can also be chosen in the form $w = tw_0t^{-1}a^i$ with $w_0 \in X$ and $0 \leqslant |i| \leqslant 1$. If $i = m$, then

$$u =_G w\gamma = tw_0t^{-1}a^m a^m t =_G w(r-2)a^m,$$

and if $i = -m$, then

$$u =_G tw_0t^{-1}a^{-m}a^m t = w(r-2),$$

both contradicting the geodesic length r of u. Then $i = 0$ and $w = tw_0t^{-1}$ with w_0 in X. We also have $\sigma_t(u) = \sigma_t(\gamma) = 1$, and Lemma 3.1 implies that $u \in PX$, and so we have the normal form $u = a^j tu_0$ with u_0 in X and $|j| \leqslant 1$.

If $j = 0$ then w and u have a common t prefix, and the path $\delta := tw_0^{-1}u_0$ has the required properties.

Suppose for the remainder of Case 8 that $|j| = 1$. Then either $\gamma = a^j t$ or $\gamma = a^{-j}t$; we consider these two subcases separately.

Case 8.1: $\gamma = a^j t$. Applying Lemma 2.1 to commute the subwords in parentheses with zero t-exponent-sum, we have

$$1 =_G w\gamma u^{-1} = tw_0t^{-1}(a^j)(tu_0^{-1}t^{-1})a^{-j} =_G tw_0u_0^{-1}t^{-1},$$

which yields that $w_0 =_G u_0$. By Proposition 2.3 we can replace each subword with a normal form $w_0 = u_0 = vta^k t^{-p}$ such that $2 \leqslant |k| \leqslant 3$ and $v \in P$ with $\sigma_t(v) = p - 1$. Since $r > 200$, Lemma 3.1 implies that $p > 10$. Let $s := \mathrm{sign}(k)$. Then $w = tvta^{|k|s}t^{-(p+1)}$ and $u = a^j tvta^{|k|s}t^{-p}$.

Consider the path $\delta := t^p a^{-2s} t^{-p} a^j t^p a^{2s} t^{1-p}$ starting at $\overline{w}$. Using Lemma 2.1 we have

$$\delta = (t^p a^{-2s} t^{-p})(a^j)t^p a^{2s} t^{-(p-1)} =_G (a^j)(t^p a^{-2s} t^{-p})t^p a^{2s} t^{-(p-1)} =_G a^j t = \gamma,$$

and so δ labels a path from $\overline{w}$ to $\bar{u}$. This path δ both follows and 'fellow travels' suffixes of w and u; see Figure 4 for a view of this path, shown in shading, when k and j have the same sign.

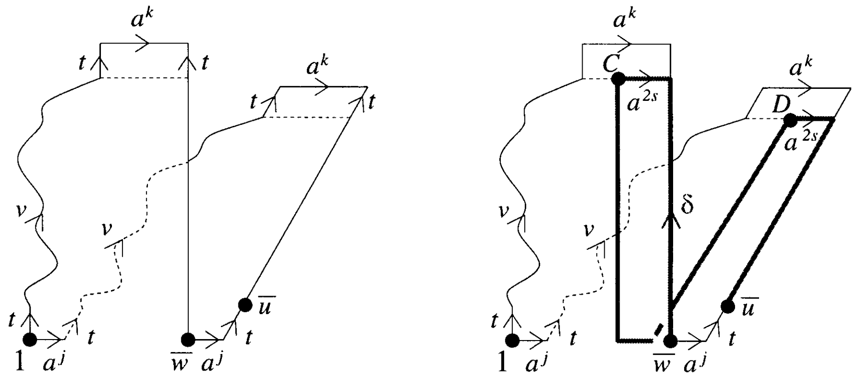

Figure 4. Case 8.1. $w = tvta^k t^{-p-1}$, $u = a^j tvta^k t^{-p}$, $\gamma = a^j t$

In order to check that δ remains in the ball $B(r)$, we analyze the distances from 1 of several vertices along the path δ, together with the lengths of the subpaths between the vertices. The prefix t^p of δ is the inverse of a suffix of w, and so starting from $\overline{w}$ the path δ follows the path w backward. Then $d(1, \overline{w\delta(i)}) = r - i$ for $0 \leqslant i \leqslant p$ and $w\delta(p) =_G w(r-p)$. The point $\overline{w\delta(p+1)}$ must then also lie in the ball $B(r-(p-1))$. Since

$$w\delta(p+2) = wt^p a^{-2s} =_G w(r-p)a^{-2s}$$
$$=_G w(r-p)ta^{-s}t^{-1} =_G w(r-(p+2))t^{-1},$$

the point $C := \overline{w\delta(p+2)}$ must lie in the ball $B(r-(p+1))$. Then the initial segment of δ of length $p+2$ from $\overline{w}$ to C lies inside $B(r)$.

Similarly, the suffix $t^{-(p-1)}$ of δ is also a suffix of u, hence

$$d(1, \overline{w}\overline{\delta(3p+5+i)}) = r-(p-1)+i$$

for $0 \leqslant i \leqslant p-1$ and $w\delta(3p+5) =_G u(r-(p-1))$. We have

$$\overline{w\delta(3p+4)} \in B(r-(p-2)).$$

Since

$$w\delta(3p+3) =_G w\delta(3p+5)a^{-2s} =_G u(r-(p-1))ta^{-s}t^{-1}$$
$$=_G a^j tvta^{|k|s}t^{-1}ta^{-s}t^{-1} =_G u(r-(p+1))t^{-1},$$

the point $D := \overline{w\delta(3p+3)}$ must lie in the ball $B(r-p)$. So the final segment of δ of length $p+1$ from D to $\bar{u}$ also lies in $B(r)$.

Finally, the central section labeled $t^{-p}a^j t^p$ of the path δ from $C \in B(r-(p+1))$ to $D \in B(r-p)$ has length $2p+1$, and hence never leaves the ball $B(r)$. The entire path δ has length $4p+4$, whereas

$$r = l(v) + |k| + p + 3 \geqslant (p-1) + 2 + p + 3 = 2p + 4,$$

and so $l(\delta) \leqslant 2r - 4$. Thus the path δ has the required properties in this subcase.

Case 8.2: $\gamma = a^{-j}t$. Applying Lemma 2.1 again yields that

$$\begin{aligned} 1 &=_G w\gamma u^{-1} = tw_0t^{-1}(a^{-j})(tu_0^{-1}t^{-1})a^{-j} \\ &=_G tw_0t^{-1}tu_0^{-1}t^{-1}a^{-j}a^{-j} =_G tw_0u_0^{-1}a^{-j}t^{-1}, \end{aligned}$$

implying that $u_0 =_G a^{-j}w_0$. Substituting this into the expression for u gives

$$u = a^j tu_0 =_G a^j ta^{-j}w_0 =_G a^{-j}tw_0.$$

Note that $r = l(w) = l(w_0) + 2$, and so $a^{-j}tw_0$ is another geodesic from 1 to $\bar{u}$. Replacing u with $a^{-j}tw_0$, we can now find the path δ using Subcase 8.1.

Case 9: *w is in class* (2) *and u is in class* (4). Then $w \in (X)N$ and u is in either NP, XNP or NPX. Since $w \notin (X)NP \cup NPX$, Lemma 3.4 says that $\sigma_t(w) < \sigma_t(u)$. Therefore $\sigma_t(\gamma) > 0$, and so $\gamma \in \{t, t^2, a^{\pm 1}t, ta^{\pm 1}\}$. We divide this case into two subcases, depending on the t-exponent sum of u.

Case 9.1: $\sigma_t(u) \leqslant 0$. By Proposition 2.3, we have the geodesic normal form $u = u_0u_1t^{-1}a^{m_e}t^f$ with $u_0 \in X \cup E$, $u_1 \in N$, $|m_e| = 1$, and $1 \leqslant f \leqslant e = |\sigma_t(u_1)| + 1$.

Case 9.1.1: $\gamma \in \{t, t^2, a^{\pm 1}t\}$. Then γ and u share a suffix t, hence $u(r-1) = u\gamma^{-1}(1)$. The geodesic w is not equal to $u(r-1)$, and so $\gamma \neq t$. For $\gamma \in \{t^2, a^{\pm 1}t\}$, the path $\delta := \gamma$ has the required properties.

Case 9.1.2: $\gamma \in \{ta^{\pm 1}\}$. Write $\gamma = ta^k$ with $|k| = 1$. Also write $w = w_0w_1$ with $w_0 \in X \cup E$ and $w_1 \in N$. Then

$$\begin{aligned} 1 &=_G \tilde{u}\gamma^{-1}\tilde{w}^{-1} = \tilde{u}_0u_1t^{-1}a^{m_e}t^f a^{-k}t^{-1}w_1^{-1}\tilde{w}_0^{-1} \\ &=_G \tilde{u}_0u_1t^{-1}a^{m_e}t^{f-1}a^{-2k}w_1^{-1}\tilde{w}_0^{-1} \in NP. \end{aligned}$$

Lemma 3.3 implies that the latter word must contain a non-geodesic $t^{-1}a^{2s}t$ subword. Then the first occurrence of a t must be in w_1^{-1}, and so $f = 1$. Write $u = u_0u_1t^{-1}a^{m_e}t = u_0u_1't$ with $u_1' := u_1t^{-1}a^{m_e} \in N$. Note that

$$w =_G u\gamma^{-1} =_G u_0 u_1' t a^{-k} t^{-1} =_G u(r-1)a^{-2k}.$$

Let

$$v := u_0 u_1' a^{-k} = u(r-1)a^{-k} =_G wa^k.$$

The vertex $\bar{v}$ satisfies $\bar{v} \in B(r)$. If $\bar{v} \in B(r-1)$, then the path $\delta := a^{2k}t$ from $\overline{w}$ to $\bar{u}$ satisfies $w\delta(1) =_G v$ and $w\delta(2) =_G u(r-1)$, and so δ is a path of length 3 inside $B(r)$ from $\overline{w}$ to $\bar{u}$. On the other hand, if $\bar{v} \notin B(r-1)$, then v is a length r geodesic in $(X)N$ and $w =_G va^{-k}$, and so $d(\bar{v}, \bar{u}) = 1$. Applying Case 7.3 to the geodesics v and w in class (2), we obtain a path δ' of length 4 inside $B(r)$ from $\overline{w}$ to $\bar{v}$. Let $\delta := \delta' a^k t$. Then δ is a path of length 6 from $\overline{w}$ to $\bar{u}$ inside $B(r)$.

Case 9.2: $\sigma_t(u) > 0$. Since $\sigma_t(w) < 0$ and $\sigma_t(w) + \sigma_t(\gamma) = \sigma_t(u)$ we must have $\sigma_t(w) = -1$, $\sigma_t(u) = 1$, and $\gamma = t^2$. By Proposition 2.3 and Lemma 3.1 we also have $w = w_0 t^{-1} a^i$ with $w_0 \in X$, $|i| \leqslant 1$, $w_0 = vta^k t^{-p-1}$ with $v \in P$, $\sigma_t(v) = p > 9$, and $2 \leqslant |k| \leqslant 3$. Since $u =_G w\gamma = vta^k t^{-p} t^{-2} a^i t^2$ and u has geodesic length $r = l(w) = l(v) + |k| + |i| + p + 3$, then $i \neq 0$. Using Lemma 2.1 we see that $u =_G (t^{-1}a^i t)(vta^k t^{-p-1})t$. The word $x := t^{-1}a^i tvta^k t^{-p}$ is another geodesic labeling a path from the identity to $\bar{u}$.

Let $s := \operatorname{sign}(k)$, and define the path

$$\delta := a^{-i} t^{p+1} a^{-2s} t^{-(p-1)} t^{-2} a^i t^2 t^{p-1} a^{2s} t^{-(p-1)}$$

starting at $\overline{w}$. Using Lemma 2.1 we have

$$\delta =_G a^{-i} t^{p+1} a^{-2s} t^{-(p-1)} (t^{p-1} a^{2s} t^{-(p-1)})(t^{-2} a^i t^2) =_G t^2 = \gamma,$$

and so δ labels a path from $\overline{w}$ to $\bar{u} = \bar{x}$. We also have $l(\delta) = 4p + 8$, and

$$r = l(w) = l(v) + |k| + p + 4 \geqslant 2p + 6,$$

hence $l(\delta) \leqslant 2r - 4$.

The proof that δ remains in $B(r)$ is similar to Case 8.1. In particular, note that

$$\overline{w\delta(p+2)} = \overline{w(r-(p+2))} \in B(r-(p+2)).$$

Since

$$w\delta(p+4) = (vta^k t^{-p} t^{-2} a^i)(a^{-i} t^{p+1} a^{-2s}) =_G vta^{k-s} t^{-1} =_G w(r-(p+4))t^{-1},$$

then $\overline{w\delta(p+4)} \in B(r-(p+3))$. Also

$$\overline{w\delta(3p+9)} = \overline{x(r-(p-1))} \in B(r-(p-1)).$$

Finally, since

$$w\delta(3p+7) =_G x(r-(p-1))a^{-2s} =_G t^{-1}a^i tvta^k t^{-1}a^{-2s}$$
$$=_G t^{-1}a^i tvta^{k-s}t^{-1} =_G x(r-(p+1))t^{-1},$$

we have $\overline{w\delta(3p+7)} \in B(r-p)$. Then the five successive intermediate subpaths of δ between $\overline{w}$, these four points, and $\bar{u}$ are too short to allow δ to leave $B(r)$.

Case 10: *Both w and u are in class* (4). In this case both w and u are in $(X)NP \cup NPX$. We may assume without loss of generality that $\sigma_t(w) \leqslant \sigma_t(u)$. It follows that $\sigma_t(\gamma) \geqslant 0$, so that $\gamma \in \{a^{\pm 1}, a^{\pm 2}, a^{\pm 1}t, ta^{\pm 1}, t, t^2\}$.

We divide this case into three subcases, depending on the t-exponents of w and u.

Case 10.1: $\sigma_t(w) \geqslant 0$ *and* $\sigma_t(u) \geqslant 0$. In this case Proposition 2.3 says that we have geodesic normal forms $u, w \in NP(X)$, and moreover $w = t^{-p_1}w'$ and $u = t^{-p_2}u'$ with $p_1 > 0$, $p_2 > 0$, and $w', u' \in P(X)$. Thus $w(1) = t^{-1} = u(1)$, and we may define $\delta := w'^{-1}t^{p_1-1}t^{-(p_2-1)}u'$.

Case 10.2: $\sigma_t(w) < 0$ *and* $\sigma_t(u) \leqslant 0$. In this subcase, we have normal forms $w, u \in (X)NP$, and we can write

$$w = w_0 w_1 t^{-1} a^{i_1} t^{f_1} \quad \text{and} \quad u = u_0 u_1 t^{-1} a^{i_2} t^{f_2}$$

with $w_0, u_0 \in X \cup E$, $w_1 \in N$, $u_1 \in N \cup E$, $f_1 \geqslant 1$, $f_2 \geqslant 1$, $\sigma_t(w_1) \leqslant -f_1$, $\sigma_t(u_1) \leqslant -(f_2-1)$, and $|i_1| = |i_2| = 1$. Lemma 3.4 implies that $\sigma_t(w_1) = \sigma_t(u_1)$. Then

$$\sigma_t(w\gamma) = \sigma_t(w_1) - 1 + f_1 + \sigma_t(\gamma) = \sigma_t(u) = \sigma_t(u_1) - 1 + f_2,$$

and so $f_2 = f_1 + \sigma_t(\gamma) \geqslant f_1$.

Case 10.2.1: $\gamma \in \{t, t^2, a^{\pm 1}t\}$. Since the last letter of u is t, the proof of Case 9.1.1 shows that $\gamma \neq t$, and for $\gamma \in \{t^2, a^{\pm 1}t\}$, we may define $\delta := \gamma$.

Case 10.2.2: $\gamma \in \{a^{\pm 1}\}$. Write $\gamma = a^k$ with $|k| = 1$. The word $\delta := t^{-1}a^{2k}t$ labels a path of length 4 from $\overline{w}$ to $\bar{u}$. Since both words w and u end with t, we have $w\delta(1) =_G w(r_1)$ and $w\delta(3) =_G u(r-1)$, and hence δ lies in $B(r)$.

Case 10.2.3: $\gamma \in \{ta^{\pm 1}\}$. Write $\gamma = ta^k$ with $|k| = 1$. In this subcase,

$$f_2 = f_1 + \sigma_t(\gamma) = f_1 + 1 \geqslant 2.$$

Then the word w ends with t and u ends with t^2. Now

$$u = u(r-1)t =_G w\gamma = wta^k =_G wa^{2k}t.$$

Let $v := u(r-1)a^{-k} =_G wa^k$. Then $v \in (X)NP$ and $\bar{v} \in B(r)$. The remainder of the proof in this subcase is similar to Case 9.1.2. If $v \in B(r-1)$, then $\delta := a^{2k}t$ has the required properties. If $v \notin B(r-1)$, then Case 10.2.2 provides a path $\delta' = t^{-1}a^{2k}t$ inside $B(r)$ from $\overline{w}$ to $\bar{v}$, and the path $\delta := \delta' a^k t = t^{-1}a^{2k}ta^k t$ from $\overline{w}$ to $\bar{u}$ satisfies the required conditions.

Case 10.2.4: $\gamma \in \{a^{\pm 2}\}$. Write $\gamma = a^{2k}$ with $|k| = 1$. In this case we have $\sigma_t(u) = \sigma_t(w) < 0$ and $f_2 = f_1 + \sigma_t(\gamma) = f_1$.

Note that the radius r satisfies

$$r = l(w) = l(w_0) + l(w_1) + 2 + f_1 \geqslant \sigma_t(w_1) + 2 + f_1 \geqslant 2f_1 + 2.$$

If $r = 2f_1 + 2$, then $w = t^{-f_1-1}a^{i_1}t^{f_1}$ and $u = t^{-f_1-1}a^{i_2}t^{f_1}$. Since $\overline{w} \neq \bar{u}$ we have $i_1 \neq i_2$, and so $i_2 = -i_1$. Since $r > 200$, then $f_1 > 1$. Now

$$u^{-1}w\gamma =_G (t^{-f_1-1}a^{i_2}t^{f_1})^{-1}(t^{-f_1-1}a^{i_1}t^{f_1})a^{2k} =_G (t^{-(f_1-1)}a^{i_1}t^{f_1-1})a^{2k};$$

according to Britton's Lemma, this last expression cannot equal the trivial element 1 in G. Thus $r \neq 2f_1 + 2$, and so $r \geqslant 2f_1 + 3$.

Define $\delta := t^{-f_1}(a^{-i_1})(t^{f_1}a^{2k}t^{-f_1})a^{i_1}t^{f_1}$. Using Lemma 2.1 to commute the subwords in parentheses, and freely reducing the resulting word, shows that $\delta =_G a^{2k} = \gamma$, and so δ labels a path from $\overline{w}$ to $\bar{u}$. We have

$$l(\gamma) = 4f_1 + 4 = 2(2f_1 + 3) - 2 \leqslant 2r - 2.$$

The prefix $\delta(f_1 + 2) = w^{-1}(f_1 + 2)$ is the inverse of a suffix of w, hence

$$\overline{w\delta(f_1 + 2)} = \overline{w(r - (f_1 + 2))} \in B(r - (f_1 + 2)).$$

The word t^{f_i} is a suffix of both δ and u, and so $\overline{w\delta(3f_1 + 4)} = \overline{u(r - f_1)} \in B(r - f_1)$. The three subpaths of δ between $\overline{w}$, the two points above, and $\bar{u}$ are again too short to allow δ to leave $B(r)$.

Case 10.3: $\sigma_t(w) < 0$ *and* $\sigma_t(u) > 0$. Then $\gamma = t^2$, $\sigma_t(w) = -1$, and $\sigma_t(u) = 1$. From Proposition 2.3, we have the normal form $w = w_0 t^{-1}a^{m_1}t^{-1}\dots t^{-1}a^{m_p}t^{p-1}$ with either $w_0 \in X$ or $w_0 = a^k \in E$ for some $|k| \leqslant 3$, $p \geqslant 2$, and $|m_p| = 1$. By Lemma 2.1, the word $\check{w} := w_0 t^{-p}a^{m_p}ta^{m_{p-1}}\dots ta^{m_1}$ is another geodesic representative of $\overline{w}$. The normal form for $u \in NP(X)$ has the form $u = t^{-e}a^j tu_1u_0$ with $e \geqslant 1$, $|j| = 1$, $u_1 \in P$ with $\sigma_t(u_1) = e$, and $u_0 \in X \cup E$. Lemma 3.4 shows that $p = e$. Replacing w by the alternate normal form $\check{w}$, we can write

$$w = w_0 t^{-p}a^i tw_2 \quad \text{and} \quad u = t^{-p}a^j tu_1u_0$$

such that either $w_0 \in X$ or $w_0 = a^k$ for $|k| \leqslant 3$, $p \geqslant 2$, $|i| = 1$, $w_2 \in P \cup E$ with $\sigma_t(w_2) = p - 2$, $|j| = 1$, $u_1 \in P$ with $\sigma_t(u_1) = p$, and $u_0 \in X \cup E$.

We will divide Case 10.3 into further subcases, depending on the form of w_0 and the length of w_2.

Case 10.3.1: $w_0 \in X$. In this case Proposition 2.3 says that we can write $w_0 = w_3 t a^k t^{-l}$ with $l \geqslant 1$, $w_3 \in P \cup E$, $\sigma_t(w_3) = l - 1$, and $2 \leqslant |m| \leqslant 3$. Let $s = \pm 1$ be the sign of m, so that $m = |m|s$. Then

$$w = w_3 t a^{|m|s} t^{-l} t^{-p} a^i t w_2.$$

The radius satisfies

$$r = l(w_3) + l(w_2) + |m| + l + p + 3 \geqslant \sigma_t(w_3) + l(w_2) + l + p + 5 = l(w_2) + 2l + p + 4.$$

Applying Lemma 2.1, we obtain that

$$u =_G \check{w} t^2 = (w_3 t a^{|m|s} t^{-l})(t^{-p} a^i t w_2 t) t =_G t^{-p} a^i t w_2 t w_3 t a^{|m|s} t^{-(l-1)}.$$

Then

$$\check{u} := t^{-p} a^i t w_2 t w_3 t a^{|m|s} t^{-(l-1)}$$

is another geodesic representative of $\bar{u}$.

Case 10.3.1.1: $l \geqslant 2$. Define

$$\delta := (w_2^{-1} t^{-1} a^{-i} t^{p-1})(t^l a^{-2m} t^{-l}) t^{-(p-1)} a^i t w_2 t^l a^{2m} t^{-(l-2)}.$$

Applying Lemma 2.1 to the subwords in parentheses shows that $\delta =_G t^2 = \gamma$, and so δ labels a path from $\overline{w}$ to $\bar{\check{u}} = \bar{u}$. The length of δ satisfies

$$l(\delta) = 2l(w_2) + 4l + 2p + 4 \leqslant 2r - 4.$$

We consider 4 vertices along the path δ. Note that

$$\overline{w\delta(l(w_2) + p + l + 1)} = \overline{w(r - (l(w_2) + p + l + 1))} \in B(r - (l(w_2) + p + l + 1)).$$

Moreover

$$\begin{aligned} w\delta(l(w_2) + p + l + 3) &=_G w_3 t a^{|m|s} t^{-1} a^{-2s} \\ &=_G w_3 t a^{(|m|-1)s} t^{-1} w(r - (l(w_2) + p + l + 3)) t^{-1}, \end{aligned}$$

implying that $\overline{w\delta(l(w_2) + p + l + 3))} \in B(r - (l(w_2) + p + l + 2))$. The suffix $t^{-(l-2)}$ of δ is also a suffix of $\check{u}$, and so

$$\overline{w\delta(2l(w_2)+3l+2p+6)} = \overline{\check{u}(r-(l-2))} \in B(r-(l-2)).$$

Finally,

$$\begin{aligned} w\delta(2l(w_2)+3l+2p+4) &=_G \check{u}(r-(l-2))a^{-2s} =_G t^{-p}a^i t w_2 t w_3 t a^{|m|s} t^{-1} a^{-2s} \\ &=_G t^{-p}a^i t w_2 t w_3 t a^{(|m|-1)s} t^{-1}, \end{aligned}$$

and so

$$\overline{w\delta(2l(w_2)+3l+2p+4)} = \overline{\check{u}(r-l)t^{-1}} \in B(r-(l-1)).$$

The five subpaths of δ between $\overline{w}$, these four points, and $\bar{u}$ are each too short to leave $B(r)$.

Case 10.3.1.2: $l = 1$. In this case define

$$\delta := (w_2^{-1}t^{-1}a^{-i}t^{p-1})(ta^{-2s}t^{-1})t^{-(p-1)}a^i t w_2 t^2 a^s.$$

Commuting the subwords in parentheses, we see that $\delta =_G t^2 = \gamma$ and δ labels a path from $\overline{w}$ to $\bar{u}$. The length satisfies

$$l(\delta) = 2l(w_2) + 2p + 9 = 2l(w_2) + 4l + 2p + 5 \leqslant 2r - 3.$$

The proof that δ remains in $B(r)$ is similar to Case 10.3.1.1. In particular,

$$\begin{aligned} \overline{w\delta(l(w_2)+p+2)} &= \overline{w(r-(l(w_2)+p+2))} \in B(r-(r-(l(w_2)+p+2)), \\ \overline{w\delta(l(w_2)+p+4)} &= \overline{w(r-(l(w_2)+p+4))t^{-1}} \in B(r-(r-(l(w_2)+p+3)), \\ \overline{w\delta(2l(w_2)+2p+8)} &= \overline{\check{u}(r-1)} \in B(r-1), \end{aligned}$$

and the four successive subpaths between $\overline{w}$, these three points, and $\bar{u}$ are too short for δ to leave $B(r)$.

Case 10.3.2: $w_0 = a^k$ *with* $|k| \leqslant 3$, *and* $l(w_2) = p - 2$. Since $\sigma_t(w_2) = p - 2$, then $w_2 = t^{p-2}$ and $w = a^k t^{-p} a^i t^{p-1}$ with $p \geqslant 2$ and $|i| = 1$. Recall that $u = t^{-p}a^j t u_1 u_0$ with $|j| = 1$ and $\sigma_t(u_1) = p$. The radius satisfies $r = |k| + 2p = p + 2 + l(u_1u_0)$, and so $|k| = l(u_1u_0) - p + 2 \geqslant 2$.

If $|k| = 2$, then $l(u_1u_0) = p$ and

$$u = t^{-p}a^j t^{p+1}.$$

We have $1 =_G wt^2u^{-1} = a^k t^{-p} a^i t^{p-1} t^2 t^{-(p+1)} a^{-j} t^p =_G a^k t^{-p} a^i a^{-j} t^p$. Since $a^k \neq_G 1$, then $i \neq j$. But $a^k t^{-p} a^i a^i t^p =_G a^k t^{-(p-1)} a^i t^{(p-1)}$, and Britton's Lemma says that the latter word cannot represent the trivial element 1, and so $i \neq -j$. Therefore we cannot have $|k| = 2$.

If $|k| = 3$, then $l(u_1u_0) = p + 1$ and $\sigma_t(u_1) = p$, and so the word u_1u_0 contains one occurrence of a or a^{-1}. Write $u_1u_0 = t^b a^l t^{p-b}$ for some $0 \leqslant b \leqslant p$ and $|l| = 1$. Then wt^2u^{-1} freely reduces to $a^k t^{-p} a^i t(t^b a^{-l} t^{-b})(a^{-j})t^p$. Commuting the subwords in parentheses and reducing again gives

$$1 =_G wt^2u^{-1} =_G a^k t^{-p} a^i t a^{-j} t^b a^{-l} t^{p-b}.$$

This word is in NP, and does not contain a subword of the form $t^{-1}a^{2m}t$, and so Britton's Lemma (or Lemma 3.3) implies a contradiction again. Therefore Case 10.3.2 cannot occur.

Case 10.3.3: $w_0 = a^k$ *with* $|k| \leqslant 3$ *and* $l(w_2) \geqslant p - 1$. In this case we have the geodesic normal forms $w = a^k t^{-p} a^i t w_2$ and $u = t^{-p} a^j t u_1 u_0$. The radius satisfies

$$r = l(w_2) + |k| + p + 2 = l(u_1u_0) + p + 2.$$

We consider two subcases, depending on whether $i = j$ or $i \neq j$.

Case 10.3.3.1: $i = j$. In this subcase note that

$$\gamma = t^2 =_G w^{-1}u = w_2^{-1} t^{-1}(a^{-j})(t^p a^{-k} t^{-p}) a^j t u_1 u_0.$$

Applying Lemma 2.1 and reducing shows that

$$\delta := w_2^{-1} t^{p-1} a^{-k} t^{-(p-1)} u_1 u_0 =_G \gamma,$$

and so δ labels a path from $\overline{w}$ to $\bar{u}$. The length of δ satisfies $l(\delta) = 2r - 6$.

As usual we analyze the distances from 1 of several vertices along the path δ. We have

$$\overline{w\delta(l(w_2))} = \overline{w(r - l(w_2))} \in B(r - l(w_2)),$$
$$\overline{w\delta(l(w_2) + 2p - 2 + |k|)} = \overline{u(r - l(u_1u_0))} \in B(r - l(u_1u_0)),$$

and the three intervening subpaths are each too short to allow δ to leave $B(r)$.

Case 10.3.3.2: $i = -j$. As in Case 10.3.3.1, after commuting and reduction we have $\gamma =_G w^{-1}u =_G w_2^{-1} t^{p-1} a^{-k} t^{-p} a^j a^j t u_1 u_0$. Then the word $\delta := w_2^{-1} t^{p-1} w_0^{-1} t^{-(p-1)} a^j u_1 u_0 =_G \gamma$ labels a path from $\overline{w}$ to $\bar{u}$ and has length $l(\delta) = 2r - 5$.

Also as in Case 10.3.3.1, we have

$$\overline{w\delta(l(w_2))} = \overline{w(r - l(w_2))} \in B(r - l(w_2))$$

and

$$\overline{w\delta(l(w_2) + 2p - 1 + |k|)} = \overline{u(r - l(u_1 u_0))} \in B(r - l(u_1 u_0)).$$

Now

$$w\delta(l(w_2) + 2p - 2 + |k|) =_G (t^{-p} a^j t) a^{-j} =_G t^{-p} a^{-j} t,$$

and so

$$\overline{w\delta(l(w_2) + 2p - 2 + |k|)} = \overline{u(r - (l(u_1 u_0) + 2)) a^{-j} t} \in B(r - l(u_1 u_0))$$

as well. The four successive subpaths of δ between $\overline{w}$, these three vertices, and $\bar{u}$ have lengths too short to allow δ to leave $B(r)$. Therefore δ has the required properties.

Therefore in each of Cases 1–10, either the case cannot occur or the path δ with the required properties can be constructed, completing the proof of Theorem 3.5.

4 Non-convexity properties for BS(1, *q*)

In the first half of this section we show, in Theorem 4.3, that the group

$$G := \mathrm{BS}(1, 2) = \langle a, t \mid tat^{-1} = a^2 \rangle$$

with generators $A := \{a, a^{-1}, t, t^{-1}\}$ does not satisfy Poénaru's $P(2)$ almost convexity condition. We start by defining some notation. Let n be a natural number with $n > 100$ and let $w := t^n a^2 t^{-n}$ and $u := at^n a^2 t^{-(n-1)}$ (see Figure 5). Then w and u are words of length $R := 2n + 2$. Moreover, using Lemma 2.1, we have $wat = (a)(t^n a^2 t^{-n})t =_G u$, and so $d(\overline{w}, \bar{u}) = 2$ and the word $\gamma := at$ labels a path from $\overline{w}$ to $\bar{u}$.

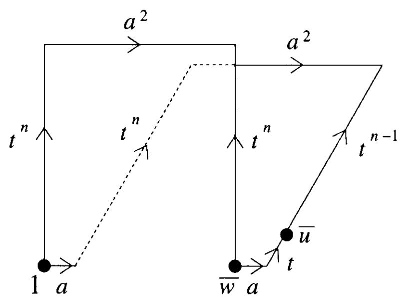

Figure 5. $w = t^n a^2 t^{-n}$, $u = at^n a^2 t^{-(n-1)}$

Lemma 4.1. *If $m \in \mathbb{Z}$ and $\overline{a^m}$ is in the ball $B(R) = B(2n + 2)$ in the Cayley graph of G, then either $m = 2^{n+1}$ or $m \leqslant 2^n + 2^{n-1} + 2^{n-2}$.*

Proof. For $\overline{a^m} \in B(R)$, Proposition 2.3 says that there is a geodesic word v in the normal form $v = t^h a^s t^{-1} a^{k_{h-1}} t^{-1} \dots t^{-1} a^{k_0}$ with $2 \leqslant |s| \leqslant 3$ and each $|k_i| \leqslant 1$, such that $v =_G a^m$. This word contains $2h$ letters of the form $t^{\pm 1}$ and $l(v) \leqslant 2n+2$, hence $h \leqslant n$. Also $v =_G a^{2^h s + 2^{h-1} k_{h-1} + \dots + k_0} =_G a^m$, and so $m = 2^h s + 2^{h-1} k_{h-1} + \dots + k_0$.

If $h = n$, then $v = t^n a^{\pm 2} t^{-n}$, and so $m = \pm 2^{n+1}$. If $h = n-1$, then there are at most four occurrences of $a^{\pm 1}$ in the expression for v; that is, $|s| + \sum |k_i| \leqslant 4$. The value of m will be maximized if $s = +3$, $k_{h-1} = +1$, and $k_i = 0$ for all $i \leqslant h-2$; in this case, $m = (3)2^{n-1} + 2^{n-2} = 2^n + 2^{n-1} + 2^{n-2}$.

Finally, if $h \leqslant n-2$, then

$$m \leqslant 2^{n-2}(3) + 2^{n-3}(1) + \dots + (1) = \frac{2^n - 1}{2 - 1},$$

and so $m < 2^n + 2^{n-1} + 2^{n-2}$.

Lemma 4.2. *The words $w = t^n a^2 t^{-n}, wa, wa^{-1}, wt^{-1}$ and $u = at^n a^2 t^{-(n-1)}$ label geodesics in the Cayley graph of G.*

Proof. As a consequence of Lemma 4.1, the vertices $\overline{a^{2^{n+1}+1}}$ and $\overline{a^{2^{n+1}-1}}$ are not in the ball $B(R)$. The words $t^n a^2 t^{-n} a = wa$ and aw both label paths from the identity to $\overline{a^{2^{n+1}+1}}$, and the word wa^{-1} labels a path from 1 to $\overline{a^{2^{n+1}-1}}$. Each of these words has length $2n+3 = R+1$, and so all three paths must be geodesic. As a consequence, the subwords w of wa and u of aw are also geodesic.

The element $\overline{wt^{-1}}$ has a geodesic normal form from Proposition 2.3 given by

$$v = t^h a^s t^{-1} a^{k_{h-1}} t^{-1} \dots t^{-1} a^{k_0} t^{-1} a^l$$

with $|l| \leqslant 1$. Since $a^{2^{n+1}} t^{-1} v^{-1} =_G 1$, Lemma 3.3 shows that $a^l = a^{2^i}$ with $i \in \mathbb{Z}$, and thus $l = 0$. Hence wt^{-1} is also geodesic.

Theorem 4.3. *The group $G = \mathrm{BS}(1,2) = \langle a, t \mid tat^{-1} = a^2 \rangle$ is not $P(2)$ with respect to the generating set $A = \{a, a^{-1}, t, t^{-1}\}$.*

Proof. Let $n \in \mathbb{N}$ with $n > 100$, $w = t^n a^2 t^{-n}$, $u = at^n a^2 t^{-(n-1)}$, and $R = 2n+2$. Then $\overline{w}$ and $\bar{u}$ lie in the sphere $S(R)$ and $d(\overline{w}, \bar{u}) = 2$. Let δ be a path inside the ball $B(R)$ from $\overline{w}$ to $\bar{u}$ that has minimal possible length. In particular, δ does not have any subpaths that traverse a single vertex more than once.

From Lemma 4.2, $\overline{wa^{\pm 1}}$ and $\overline{wt^{-1}}$ are not in $B(R)$, and so the first letter of the path δ must be t. Let $\pi : \mathscr{C} \to T$ denote the horizontal projection map from the Cayley complex of G to the regular tree T of valence 3, as described at the beginning of Section 2. The vertices $\pi(\overline{w\delta(1)}) = \pi(\bar{t})$ and $\pi(\bar{u}) = \pi(\overline{at})$ are the terminal vertices of the two distinct edges of T with initial vertex $\pi(\overline{w}) = \pi(1)$. Since the projection of the path δ begins at $\pi(1)$, goes to $\pi(\bar{t})$, and eventually ends at $\pi(\overline{at})$, there must be another point $P := \overline{w\delta(j)}$ along the path δ with $\pi(P) = \pi(\overline{w\delta(j)}) = \pi(1)$ and $1 < j < l(\delta)$. Let δ_1 be the subpath of δ from $\overline{w}$ to P.

Our assumption that δ has minimal possible length implies that $P \neq \overline{w}$. Since $\pi(P) = \pi(1)$, we have $P =_G a^m$ for some $m \in \mathbb{Z}$. Then Lemma 4.1 shows that $m \leqslant 2^n + 2^{n-1} + 2^{n-2}$. Since δ_1 labels a path from $\overline{a^{2^{n+1}}}$ to $\overline{a^m}$, we have $\delta_1^{-1} =_G a^{2^{n+1}-m} = a^k$ with $k = 2^{n+1} - m \geqslant 2^{n-2} > 2^{(n-4)+1}$. Applying the contrapositive of Lemma 4.1, we conclude that $\overline{a^k}$ is not in the ball $B(2(n-4)+2)$, hence $l(\delta_1^{-1}) > 2(n-4)+2$. Therefore $l(\delta) > R - 8$, and so this length cannot be bounded above by a sublinear function of R.

For the remainder of this section, let $G_q := \mathrm{BS}(1,q) = \langle a, t \mid tat^{-1} = a^q \rangle$ with $q \geqslant 7$ and with generators $A := \{a, a^{-1}, t, t^{-1}\}$. We will apply methods very similar to those developed above, to show that these groups are not MAC.

Lemma 4.4. *If $m \in \mathbb{Z}$ and $\overline{a^m}$ is in the ball $B(R) = B(2n+1)$ in the Cayley graph of G_q, then either $m = q^n$ or $m \leqslant 3q^{n-1}$.*

Proof. For $\overline{a^m} \in B(R)$, Proposition 2.3 says that there is a geodesic word v in the normal form $v = t^h a^s t^{-1} a^{k_{h-1}} t^{-1} \dots t^{-1} a^{k_0}$ with $1 \leqslant |s| \leqslant q-1$ and $|k_i| \leqslant \lfloor \frac{q}{2} \rfloor$ for each i, such that $v =_G a^m$. Then $h \leqslant n$ and $m = q^h s + q^{h-1} k_{h-1} + \cdots + k_0$.

If $h = n$, then $v = t^n a^{\pm 1} t^{-n}$, and $m = \pm q^n$. If $h = n-1$, then there are at most 3 occurrences of $a^{\pm 1}$ in the expression for v. The value of m will be maximized if $s = +3$, in which case $m = (3)q^{n-1}$. Finally, if $h \leqslant n-2$, then

$$m \leqslant q^{n-2}(q-1) + q^{n-3} \lfloor \tfrac{q}{2} \rfloor + \cdots + \left(\lfloor \tfrac{q}{2} \rfloor \right) = q^{n-1} - q^{n-2} + \frac{q^{n-2} - 1}{2 - 1} \lfloor \tfrac{q}{2} \rfloor,$$

and so $m < 3q^{n-1}$.

Theorem 4.5. *The group $G_q = \mathrm{BS}(1,q) = \langle a, t \mid tat^{-1} = a^q \rangle$ with $q \geqslant 7$ is not* MAC *with respect to the generating set $A = \{a, a^{-1}, t, t^{-1}\}$.*

Proof. Let n be a natural number with $n > 100$. Let $w' := t^n a t^{-n}$ and $u' := a t^n a t^{-(n-1)}$. Then w' and u' are words of length $R := 2n+1$. As a consequence of Lemma 4.4, an argument similar to the proof of Lemma 4.2 shows that the words $w', u', w'a^{\pm 1}$, and $w't^{-1}$ are geodesics. Hence $\overline{w'}$ and $\overline{u'}$ lie in $S(R)$. Lemma 2.1 says that $w'at = (a)(t^n a t^{-n})t =_G u'$, so that $d(\overline{w'}, \overline{u'}) = 2$ and the word $\gamma := at$ labels a path from $\overline{w'}$ to $\overline{u'}$. Let δ be a path inside the ball $B(R)$ from $\overline{w'}$ to $\overline{u'}$ that has minimal possible length.

From the information on geodesics in the previous paragraph, the first letter of the path δ must be t. Let $\pi : \mathscr{C} \to T$ denote the horizontal projection map from the Cayley complex of G_q to the regular tree T of valence $q+1$. The vertices $\pi(\overline{w'\delta(1)}) = \pi(\overline{t})$ and $\pi(\overline{u'}) = \pi(\overline{at})$ are the terminal vertices of two distinct edges of T with initial vertex $\pi(\overline{w'}) = \pi(1)$. Consequently, there must be a point $P := \overline{w'\delta(j)}$ along the path δ with $\pi(P) = \pi(\overline{w'\delta(j)}) = \pi(1)$ and $1 < j < l(\delta)$. Write $\delta = \delta_1 \delta_2$ where δ_1 is the subpath of δ from $\overline{w'}$ to P.

We have $P \neq \overline{w'} = \overline{a^{q^n}}$, and $P =_{G_q} a^m$ for some $m \in \mathbb{Z}$, hence Lemma 4.4 shows that $m \leqslant 3q^{n-1}$. The word δ_1 labels a path from $\overline{a^{q^n}}$ to $\overline{a^m}$, and so $\delta_1 =_{G_q} a^k$ with $k = q^n - m \geqslant (q-3)q^{n-1} > 3q^{n-1}$ since $q \geqslant 7$. Then Lemma 4.4 says that either $k = q^n$ or $\overline{a^k}$ is not in the ball $B(2n+1)$.

If $\overline{a^k}$ is not in the ball $B(2n+1)$, then $l(\delta_1) > 2n+1 = R$. Note that $u't^{-1} = aw' =_{G_q} a^{q^n+1}$. The word $\delta_2 t^{-1}$ labels a path from $P = \overline{a^m}$ to $\overline{a^{q^n+1}}$, and so $\delta_2 t^{-1} =_{G_q} a^{k+1}$ with $k+1 > 3q^{n-1}$. Then $l(\delta_2 t^{-1}) > R$ as well. Thus $l(\delta_2) \geqslant R$ and $l(\delta) \geqslant R + 1 + R = 2R+1$. Since there is a path $w'^{-1}u'$ of length $2R$ inside $B(R)$ from $\overline{w'}$ to $\overline{u'}$, this contradicts our choice of δ with minimal length.

Then the path δ satisfies $k = q^n$, so that $P = 1$. Therefore δ reaches the vertex corresponding to the identity, and the length of δ is $2R$.

Corollary 4.6. *The properties* MAC *and* M′AC *are not commensurability invariant, and hence also not quasi-isometry invariant.*

Proof. The subgroup of index 3 in $\mathrm{BS}(1,2) = \langle a, t \mid tat^{-1} = a^2 \rangle$ generated by a and t^3 is isomorphic to $\mathrm{BS}(1,8)$. Theorem 3.5 shows that $\mathrm{BS}(1,2)$ is M′AC and hence MAC, and Theorem 4.5 proves that $\mathrm{BS}(1,8)$ has neither property.

5 Stallings' group is not MAC

In [15], Stallings showed that the group with finite presentation

$$S := \langle a, b, c, d, s \mid [a,c] = [a,d] = [b,c] = [b,d] = 1,$$
$$(a^{-1}b)^s = a^{-1}b, (a^{-1}c)^s = a^{-1}c, (a^{-1}d)^s = a^{-1}d \rangle$$

does not have homological type FP_3. In our notation, $[a,c] := aca^{-1}c^{-1}$ and $(a^{-1}b)^s := sa^{-1}bs^{-1}$. Let $X := \{a, b, c, d, s, a^{-1}, b^{-1}, c^{-1}, d^{-1}, s^{-1}\}$ be the inverse closed generating set, and let Γ be the corresponding Cayley graph of S.

Let G be the subgroup of S generated by $Y := \{a, b, c, d, a^{-1}, b^{-1}, c^{-1}, d^{-1}\}$, and let Λ be the corresponding Cayley graph of G. Then G is the direct product of the non-abelian free groups $\langle a, b \rangle$ and $\langle c, d \rangle$. Let H be the finitely generated subgroup of G given by $H = \langle a^{-1}b, a^{-1}c, a^{-1}d \rangle$.

Lemma 5.1. *The group H consists of all elements of G that can be represented by a word over Y of exponent sum zero. Moreover, every word, over X or Y, representing an element of H must have exponent sum zero.*

Proof. Using the fact that a commutes with both c and d, we have $ca^{-1}, da^{-1} \in H$, and so $ac^{-1}, ad^{-1} \in H$. Using the fact that b and c commute, we have

$$ab^{-1} =_S a(c^{-1}c)b^{-1} =_S (ac^{-1})b^{-1}c =_S (ac^{-1})b^{-1}(aa^{-1})c$$
$$=_S (ac^{-1})(a^{-1}b)^{-1}(a^{-1}c) \in H$$

as well. Taking products of the form $h_1^{-1}h_2$ with

$$h_1, h_2 \in \{a^{-1}b, a^{-1}c, a^{-1}d, ab^{-1}, ac^{-1}, ad^{-1}\}$$

we see that $l_1^{-1}l_2, l_1 l_2^{-1} \in H$ for all positive letters $l_1, l_2 \in \{a, b, c, d\}$. Finally, consider an arbitrary word $w = l_1^{\varepsilon_1} \dots l_n^{\varepsilon_n}$ with $l_i \in \{a, b, c, d\}$, $\varepsilon_i = \pm 1$, and $\sum_i \varepsilon_i = 0$. For each i, there is a letter $m \in Y$ which commutes with both $l_i^{\varepsilon_i}$ and $l_{i+1}^{\varepsilon_{i+1}}$. Repeating the technique above of inserting the inverse pair mm^{-1} between $l_i^{\varepsilon_i}$ and $l_{i+1}^{\varepsilon_{i+1}}$ and applying the commutation relations as needed, we can write w as a product of elements of exponent sum zero of the form $m_1 m_2$ with $m_i \in Y$. Then $\overline{w} \in H$. The second sentence of this lemma follows from the fact that the exponent sum for each of generators of H and each of the relators in the presentation for S is zero.

Let $\phi : H \to H$ be the identity function. Then S is the HNN extension $S = G\star_\phi$ with stable letter s, and s commutes with all of the elements of H.

Lemma 5.2. *Let $w \in X^*$.*

(1) *If w is a geodesic in Γ, then the word w cannot contain a subword of the form sus^{-1} or $s^{-1}us$ with $\bar{u} \in H$.*

(2) *If $\overline{w} \in G$ and w is a geodesic in Γ, then $w \in Y^*$ and w is a geodesic in Λ.*

(3) *If $w \in Y^*$ and w is a geodesic in Λ, then w, sw and ws are all geodesics in Γ.*

Proof. Part (1) follows directly from the fact that for $\bar{u} \in H$, $sus^{-1} =_S s^{-1}us =_S u$. In parts (2) and (3), suppose that $g \in G$, v is a geodesic word over X in Γ representing g, and $w \in Y^*$ is a geodesic in Λ with $\overline{w} = g$ also. Then $vw^{-1} =_S 1$. Britton's Lemma applied to the HNN extension S says that if either s or s^{-1} occurs in vw^{-1}, then vw^{-1}, and hence v, must contain a subword of the form sus^{-1} or $s^{-1}us$ with $\bar{u} \in H$, contradicting part (1). Therefore $v \in Y^*$. Since $v, w \in Y^*$ and v is is a geodesic in Λ, then $l(v) \leqslant l(w)$ Similarly since $v, w \in X^*$ and w is a geodesic in Γ we have $l(w) \leqslant l(v)$. Thus v is also a geodesic in Λ, and w is a geodesic in Γ.

For the remainder of part (3), suppose that μ is a geodesic representative of sw in Γ. Then $w^{-1}s^{-1}\mu =_S 1$. Britton's Lemma then says that $w^{-1}s^{-1}\mu$ must contain a subword of the form sus^{-1} or $s^{-1}us$ with $\bar{u} \in H$. Since μ is a geodesic, part (1) says that sus^{-1} or $s^{-1}us$ cannot be completely contained in μ, and so we can write $\mu = \mu_1 s \mu_2$ with $\overline{\mu_1} \in H$. Since μ_1 and s commute, $s\mu_1\mu_2 =_S \mu =_S sw$, hence $\mu_1\mu_2 =_S w$ and both $\mu_1\mu_2$ and w are geodesics representing the same element of G. Hence $l(\mu_1\mu_2) = l(w)$. Then

$$l(\mu) = l(\mu_1) + 1 + l(\mu_2) = l(w) + 1 = l(sw),$$

and so sw is a geodesic in Γ. The proof that ws is also a geodesic in Γ is similar.

The proof of the following theorem relies further on the HNN extension structure of Stallings' group S. In particular, we utilize an 's-corridor' to show that the path δ in the definition of MAC cannot exist.

Theorem 5.3. (S, X) *is not* MAC *with respect to the generating set* X.

Proof. Let $\alpha := b^{-(n+1)}a^{n+1}$ and $\beta := sb^{-(n+1)}a^n$, and let $\chi = b^{-(n+1)}a^n$ be their maximal common subword. Then $\alpha \in Y^*$, and so $\bar{\alpha} \in G$; in particular, the exponent sum of α is zero, so Lemma 5.1 says that $\bar{\alpha} \in H$ also. Since α is a geodesic in the Cayley graph Λ of the group $G = F_2 \times F_2$, Lemma 5.2(3) says that α is a geodesic in Γ. Similarly, χ is a geodesic in Λ and so Lemma 5.2(3) says that $\beta = s\chi$ is also geodesic in Γ. Thus α and β lie in the sphere of radius $2n + 2$ in Γ. Since

$$\alpha^{-1}\beta = a^{-(n+1)}b^{n+1}sb^{-(n+1)}a^n =_S sa^{-(n+1)}b^{n+1}b^{-(n+1)}a^n =_S sa^{-1},$$

we have $d(\bar{\alpha}, \bar{\beta}) = 2$ for all natural numbers n.

Suppose that there is a path δ of length at most $2(2n + 2) - 1$ inside the ball of radius $2n + 2$ between $\bar{\alpha}$ and $\bar{\beta}$. Since the relators in the presentation of S have even length, the word δ must have length at most $2(2n + 2) - 2 = 4n + 2$.

Applying Britton's Lemma to the product $\delta a s^{-1} =_S 1$ shows that $\delta = w_1 s w_2$ with $\overline{w_1}, \overline{w_2 a} \in H$. Then $\overline{w_1}, \overline{w_2} \in G = F_2 \times F_2$. Lemma 5.2(2) and the direct product structure imply that there are geodesic representatives q_1 and q_2 of $\overline{w_1}$ and $\overline{w_2}$, respectively, that have the form $q_1 = q_{1_{a,b}} q_{1_{c,d}}$ and $q_2 = q_{2_{c,d}} q_{2_{a,b}}$ with $q_{1_{a,b}}, q_{2_{a,b}} \in \{a, b, a^{-1}, b^{-1}\}^*$ and $q_{1_{c,d}}, q_{2_{c,d}} \in \{c, d, c^{-1}, d^{-1}\}^*$.

Since $\bar{\alpha}$ and $\overline{q_1} = \overline{w_1}$ are both elements of H, we have $\overline{\alpha q_1} \in H$ as well. From the direct product structure, there is a geodesic representative $\sigma \in Y^*$ of $\overline{\alpha q_1}$ of the form $\sigma = \sigma_{a,b}\sigma_{c,d}$ with $\sigma_{a,b} \in \{a, b, a^{-1}, b^{-1}\}^*$ and $\sigma_{c,d} \in \{c, d, c^{-1}, d^{-1}\}^*$ (see Figure 6).

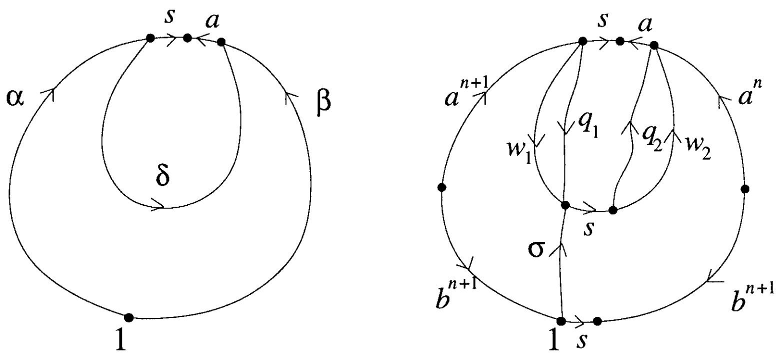

Figure 6. Paths in the Cayley graph of Stallings' group

The edge in Γ labeled by s connecting $\bar{\sigma}$ and $\overline{\sigma s}$ is part of the path δ, and so this edge must lie in the ball of radius $2n + 2$ in Γ. Lemma 5.2(3) says that σs is a geodesic, hence

$$d(1, \bar{\sigma}) + 1 = l(\sigma) + 1 = l(\sigma s) = d(1, \overline{\sigma s}) \leqslant 2n + 2.$$

Then $\bar{\sigma} \in B(2n + 1)$ and $l(\sigma) \leqslant 2n + 1$.

Now $l(q_1) + 1 + l(q_2) \leqslant l(\delta) \leqslant 4n + 2$, and thus either $l(q_1) \leqslant 2n$ or $l(q_2) \leqslant 2n$ (or both).

Case A: $l(q_1) \leqslant 2n$. Note that

$$\alpha q_1 \sigma^{-1} = b^{-(n+1)} a^{n+1} q_{1_{a,b}} q_{1c,d} \sigma_{c,d}^{-1} \sigma_{a,b}^{-1} =_{F_2 \times F_2} 1.$$

Hence $q_{1c,d} =_{F_2} \sigma_{c,d}$ and $\alpha q_{1_{a,b}} =_{F_2} \sigma_{a,b}$. Since geodesics in free groups are unique, we also have $q_{1c,d} = \sigma_{c,d}$.

There is an integer i_1 with $0 \leqslant i_1 \leqslant 2n$ such that

$$q_{1_{a,b}}(i_1) = \alpha^{-1}(i_1) \quad \text{but} \quad q_{1_{a,b}}(i_1 + 1) \neq \alpha^{-1}(i_1 + 1),$$

where we write $q_{1_{a,b}}(0) := 1$ and $q_{1_{a,b}}(k) := q$ for all $k > l(q_{1_{a,b}})$. Write $q_{1_{a,b}} = \alpha^{-1}(i_1) r$ with $r \in F_2 = \langle a, b \rangle$. The words $\alpha, q_{1_{a,b}}$, and $\sigma_{a,b}$ are all geodesic representatives of elements of the free group F_2, and hence these are freely reduced words that define non-backtracking edge paths in the tree given by the Cayley graph for this group. By definition of i_1, the product $\alpha q_{1_{a,b}}$ freely reduces to $\alpha(2n + 2 - i_1) r$, with no further free reduction possible. Then $\alpha(2n + 2 - i_1) r$ is the unique geodesic representative in $F_2 = \langle a, b \rangle$ of $\alpha q_{1_{a,b}}$, and hence $\alpha(2n + 2 - i_1) r = \sigma_{a,b}$.

Case A.1: $i_1 \leqslant n + 1$. In this case, $q_{1_{a,b}} = a^{-i_1} r$. Now $q_1 = q_{1_{a,b}} q_{1_{c,d}} = a^{-i_1} r q_{1_{c,d}}$ represents an element of H, and so Lemma 5.1 says that q_1 has exponent sum zero. Then $l(r q_{1_{c,d}}) \geqslant i_1$. We also have $\sigma = \sigma_{a,b} \sigma_{c,d} = \alpha(2n + 2 - i_1) r q_{1_{c,d}}$. Then

$$l(\sigma) \geqslant (2n + 2 - i_1) + i_1 = 2n + 2,$$

contradicting the result above that $l(\sigma) \leqslant 2n + 1$. Thus this subcase cannot occur.

Case A.2: $i_1 > n + 1$. In this case, $\sigma_{a,b} = b^{-(2n+2-i_1)} r$. Since

$$\sigma = \sigma_{a,b} \sigma_{c,d} = b^{-(2n+2-i_1)} r \sigma_{c,d}$$

represents an element of H, this word has exponent sum zero, and so $l(r\sigma_{c,d}) \geqslant 2n + 2 - i_1$ in this subcase. The word $q_1 = q_{1_{a,b}} q_{1_{c,d}} = \alpha^{-1}(i_1) r \sigma_{c,d}$ then has length

$$l(q_1) \geqslant i_1 + (2n + 2 - i_1) = 2n + 2,$$

contradicting the fact that we are in Case A.

Case B: $l(q_2) \leqslant 2n$. Since $\sigma \in H$, σ commutes with s. Then

$$\sigma =_S s^{-1} \sigma s =_S s^{-1} \alpha q_1 s =_S s^{-1} \alpha q_1 s q_2 q_2^{-1} =_S s^{-1} \beta q_2^{-1} =_S \chi q_2^{-1}.$$

In this case we have $\sigma_{a,b}\sigma_{c,d} =_{F_2\times F_2} b^{-(n+1)}a^{n+1}q_{2_{a,b}}^{-1}q_{2c,d}^{-1}$, and so $q_{2c,d}^{-1} =_{F_2} \sigma_{c,d}$ and $\chi q_{2_{a,b}}^{-1} =_{F_2} \sigma_{a,b}$. Uniqueness of geodesics in $F_2 = \langle c, d\rangle$ implies that $q_{2c,d}^{-1} = \sigma_{c,d}$.

There is an integer i_2 with $0 \leqslant i_2 \leqslant 2n$ such that

$$q_{2_{a,b}}^{-1}(i_2) = \chi^{-1}(i_2) \quad \text{but} \quad q_{2_{a,b}}^{-1}(i_2+1) \neq \chi^{-1}(i_2+1).$$

Write $q_{2_{a,b}} = r(\chi^{-1}(i_2))^{-1}$ with $r \in F_2 = \langle a, b\rangle$. The words $\chi, q_{2_{a,b}}$, and $\sigma_{a,b}$ are all geodesics, and hence freely reduced words, in F_2. By definition of i_2, the product $\chi q_{2_{a,b}}^{-1}$ freely reduces to $\chi(2n+1-i_2)r^{-1}$, with no further reduction possible. Then $\chi(2n+1-i_2)r^{-1} = \sigma_{a,b}$.

Case B.1: $i_2 \leqslant n$. In this case, $q_{2_{a,b}} = ra^{i_2}$. Now $q_2 = q_{2_{c,d}}q_{2_{a,b}} = q_{2_{c,d}}ra^{i_2}$. Recall that q_2 was chosen as a geodesic representative of an element $\overline{w_2} \in G$ for which $\overline{w_2 a} \in H$. Then $q_2 a$ represents an element of H, and so (by Lemma 5.1) has exponent sum zero. Therefore the exponent sum of q_2 is -1. Then $l(q_{2_{c,d}}r) \geqslant i_2 + 1$. We also have

$$\sigma = \sigma_{a,b}\sigma_{c,d} = \chi(2n+1-i_2)r^{-1}q_{2_{c,d}}^{-1}.$$

Then

$$l(\sigma) \geqslant (2n+1-i_2) + (i_2+1) = 2n+2,$$

again contradicting the result above that $l(\sigma) \leqslant 2n+1$.

Case B.2: $i_2 > n$. In this case, $\sigma_{a,b} = b^{-(2n+1-i_2)}r^{-1}$. Since

$$\sigma = \sigma_{a,b}\sigma_{c,d} = b^{-(2n+1-i_2)}r^{-1}\sigma_{c,d}$$

represents an element of H, this word has exponent sum zero, so that $l(r^{-1}\sigma_{c,d}) \geqslant 2n+1-i_2$ in this subcase. Therefore the word

$$q_2 = q_{2_{c,d}}q_{2_{a,b}} = \sigma_{c,d}^{-1}r(\chi^{-1}(i_1))^{-1}$$

has length $l(q_2) \geqslant (2n+1-i_2) + i_2 = 2n+1$, contradicting the fact that we are in Case B.

Therefore every subcase results in a contradiction implying that the subcase cannot occur. Then the path δ cannot exist, and so S is not MAC with respect to the generating set X.

References

[1] J. Belk and K.-U. Bux. Thompson's group F is not minimally almost convex. arXiv:math.GR/0301141.
[2] M. R. Bridson. Doubles, finiteness properties of groups, and quadratic isoperimetric inequalities. *J. Algebra* **214** (1999), 652–667.

[3] J. W. Cannon. Almost convex groups. *Geom. Dedicata* **22** (1987), 197–210.
[4] M. Elder. Automaticity, almost convexity and falsification by fellow traveler properties of some finitely generated groups. Ph.D. thesis. University of Melbourne (2000).
[5] M. Elder. The loop shortening property and almost convexity. *Geom. Dedicata* **102** (2003), 1–18.
[6] D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson and W. P. Thurston. *Word processing in groups* (Jones and Bartlett Publishers, 1992).
[7] L. Funar. On discrete solvgroups and Poénaru's condition. *Arch. Math.* (*Basel*) **72** (1999), 81–85.
[8] J. R. J. Groves. Minimal length normal forms for some soluble groups. *J. Pure Appl. Algebra* **114** (1996), 51–58.
[9] V. Guba. The Dehn function of Richard Thompson's group F is quadratic. arXiv:math.GR/0211395.
[10] I. Kapovich. A note on the Poénaru condition. *J. Group Theory* **5** (2002), 119–127.
[11] C. F. Miller, III. Normal forms for some Baumslag–Solitar groups. Preprint (1997).
[12] C. F. Miller, III and M. Shapiro. Solvable Baumslag–Solitar groups are not almost convex. *Geom. Dedicata* **72** (1998), 123–127.
[13] V. Poénaru. Almost convex groups, Lipschitz combing, and π_1^∞ for universal covering spaces of closed 3-manifolds. *J. Differential Geom.* **35** (1992), 103–130.
[14] T. R. Riley. The geometry of groups satisfying weak almost-convexity or weak geodesic-combability conditions. *J. Group Theory* **5** (2002), 513–525.
[15] J. Stallings. A finitely presented group whose 3-dimensional integral homology is not finitely generated. *Amer. J. Math.* **85** (1963), 541–543.
[16] C. Thiel. *Zur fast-Konvexität einiger nilpotenter Gruppen*. Universität Bonn Mathematisches Institut (1992). (Dissertation. Rheinische Friedrich-Wilhelms-Universität Bonn (1991).)

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